BAYESIAN ESTIMATION OF THE GJR-GARCH (p, q) MODEL WITH STUDENT-T PRIOR DISTRIBUTION

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Abstract

The presence of volatility in many financial time series data is one of the problems that cause the variance to be non-constant. The GJR-GARCH (p, q) is a model that takes into account time-varying volatility, allowing positive and negative shocks to have distinct effects. This study provides the estimates of the GJR-GARCH (p, q) model using the Bayesian approach. Student-t distribution is used as prior error distribution. It derives the posterior distribution of the GJR-GARCH (p, q) model with student-t distribution, specifically the parameters α and β, latent variable ω, and degrees of freedom v.

Keywords: *volatility, GJR-GARCH, Bayesian, student-t, posterior distribution*

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1.0 Introduction

Volatility is the frequency at which the market price of an investment increases or decreases. It is measured by calculating the standard deviation of the annualized returns over a given period of time. If the costs of a security fluctuate rapidly in a short period of time, it is termed to have high volatility. Conversely, if the prices of a security fluctuate slowly over a longer period, it is termed to have low volatility. Robert Engle in 1982 developed the Autoregressive Conditional Heteroskedastic (ARCH) model which was the first model of time-varying volatility. The ARCH model grew rapidly as a volatility forecasting technique during the last thirty years and has been applied to numerous economic and financial data series. However, in many applications with the ARCH model, a long lag length or a large number of parameters are required to approximately model the data. Thus, Engle's student, Tim Bollerslev (1986), developed the Generalized Autoregressive Conditional Heteroskedastic (GARCH) model in which there is past conditional errors aside from conditional variance as part of the model. GARCH model had the same properties as the ARCH model but required less parameters to model heteroskedasticity precisely. When using these two models, there is an imposed restriction on the parameters to assure that the variance is positive.

For this reason, Nelson (1991) presented an alternative way to the GARCH model, the Exponential GARCH (EGARCH) model, by modifying it to allow the asymmetric effect of positive and negative stock return. Another model which allows the positive and negative shocks to have different impact on the volatilities is the GJR-GARCH model, which was introduced by Glosten *et al.,* (1993). Later, many models were developed and extended regarding volatility models. These models were estimated using the Maximum Likelihood Estimation (MLE). Maximum Likelihood estimates are known to be statistically efficient, and its likelihood ratio test provides a powerful and general method of inference. However, the complexity of the computations of maximum likelihood estimates made it less practical in many situations. According to Engle (1982), the normality assumption of the error terms may not be appropriate in some applications since heavy tails are commonly observed in economic and financial data. Some study shows that the student-t provides a suitable description for this type of data. Bollerslev (1987) introduced the student-t GARCH model in which the error terms are assumed to be t-distributed and conclude that GARCH models with normal errors do not seem to fully capture leptokurtosis. This paper derived the parameters of the GJR-GARCH (p, q) model with student-t distribution error using the Bayesian estimation.

2.0 Literature Review

The GJR-GARCH model was named after the authors who introduced it (Glosten *et al.,* 1993), as an alternative way to model asymmetric effects. It extends the standard GARCH to include asymmetric terms that capture an important phenomenon in the conditional variance. Following Ardia (2008), it is assumed that the error terms can be modeled as:

$$u_t = \varepsilon_t(\sigma h_t)^{\frac{1}{2}}$$

$$\varepsilon_t \sim iid\ S_v(0,1)$$

$$\sigma = \frac{v-2}{v}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q}\left(\alpha_i I_{\{u_{t-1}\geq 0\}} + \alpha_i^* I_{\{u_{t-1}<0\}}\right)u_{t-1}^{\ 2} + \sum_{j=1}^{q}\beta_j h_{t-j'} \qquad (1)$$

where α_0>0, α_i≥0 (i=1,…,q), α_i^*≥0 (i=1,…,q) and β_j≥0(j=1,…,p) to guarantee that the conditional variance is positive. S_v (0,1) is the standard student-t density with v degrees of freedom.

Bayesian Estimation

Definition 1

If A and B are events in the sample space Ω and $P(A)$>0, then the conditional probability of B given A is

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$

In case P(A)=0, we make the convention that P(B|A)=P(B).

Theorem 1 (Bayes' Rule) Let A and B be even in the probability space (Ω,F,P) which are non-empty sets. Then the probability A given B is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Let A_1,A_2,A_3,… be a partition of the sample space and let B be any set. Then, for each i=1,2,… ,

$$P(Ai|B) = \frac{P(B|Ai)P(Ai)}{P(B|Ai)P(Ai)}$$

The Prior Distribution

In the Bayesian point of view, the parameter θ is treated as a random variable and the prior information is an essential problem for many statistical decisions. A convenient way to quantify such information is to express it as a probability distribution. In Bayesian Statistical inference, a prior probability distribution, often called prior, of an uncertain quantity is the probability distribution that would express one's beliefs about quantity before some evidence for the parameters of prior distribution are called *hyperparameters.*

Definition 2

Let X=(x_1,…,x_n) be a random sample and Θ=θ_1,…,θ_k be the parameter of interest and π(Θ) be the prior distribution associated with Θ and $f(X|\Theta)$ the distribution from which the sample was taken. Then the posterior distribution of Θ given $\boldsymbol{X}$ is defined as

$$\pi(\theta|X) = k.\pi(\theta)L(\theta|X), \tag{2}$$

where $k = \frac{1}{\int \dots \int \pi(\theta)L(\theta \mid X)d\theta}$

The likelihood function of the sample X given Θ, L(Θ|X) is a function of Θ and not of the sample X.

Proposition 1 The model given by

$$u_t = \varepsilon_t(\omega_t \sigma h_t)^{\frac{1}{2}} \text{ for } t = 1, \dots, T \tag{3}$$

$$\varepsilon_t \sim iid\ N(0,1)$$

$$\omega_t \sim iid\ IG\left(\tfrac{v}{2}, \tfrac{v}{2}\right)$$

is equivalent to the model given in equation (1) where IG denotes the Inverted Gamma density with parameters $\frac{v}{2}$ and $\frac{v}{2}$. ε_t is a sequence of independent and identically distributed random variables with $E(\varepsilon_t)=0$ and $var(\varepsilon_t)=1$.

Proof: Let the parameters of the model be Δ=(α,β,v). From equation (1), the density function of ω_t is

$$p(\omega_t \mid v) = \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\tau\left(\frac{v}{2}\right)} \omega_t^{-\frac{v}{2}-1} exp\left\{-\frac{v}{2\omega_t}\right\}.$$

We can write the joint density of T x 1 vector $\omega=(\omega_1,\omega_2,\dots,\omega_T)'$ as

$$p(\omega \mid v) = \left(\tfrac{v}{2}\right)^{\frac{Tv}{2}}\left[\tau\left(\tfrac{v}{2}\right)\right]^{-T} \prod_{t=1}^{T} \omega_t^{-\frac{v}{2}-1} exp\left\{-\frac{v}{2\omega_t}\right\} \tag{4}$$

The density function of the error term u_t in (3) is

$$f(u_t \mid \Delta, \omega) = \frac{1}{(2\pi\omega_t\sigma h_t)^{\frac{1}{2}}} exp\left\{-\frac{1}{2}\frac{u_t^2}{\omega_t\sigma h_t}\right\}.$$

Then we express the likelihood of (Δ,ω) as

$$L(\Delta,\omega \mid u) \propto \prod_{t=1}^{T} \frac{1}{(2\pi\omega_t\sigma h_t)^{\frac{1}{2}}} exp\left\{-\frac{1}{2}\frac{u_t^2}{\omega_t\sigma h_t}\right\}.$$

Using Bayes Theorem,

$$\pi(\Delta,\omega \mid u) = L(\Delta,\omega \mid u)p(\omega \mid v)$$

$$= \left(\tfrac{v}{2}\right)^{\frac{Tv}{2}}\left[\tau\left(\tfrac{v}{2}\right)\right]^{-T} \prod_{t=1}^{T} \frac{1}{(2\pi\sigma h_t)^{\frac{1}{2}}} \omega_t^{-\frac{(v+1)}{2}-1} exp\left\{-\left(\frac{u_t^2}{2\sigma h_t}+\frac{v}{2}\right)\frac{1}{w_t}\right\}$$

Integrating π(Δ,ω|u) with respect to ω where $0<\omega_t<\infty$,

$$\pi(\Delta \mid u) = \left(\tfrac{v}{2}\right)^{\frac{Tv}{2}}\left[\tau\left(\tfrac{v}{2}\right)\right]^{-T} \prod_{t=1}^{T} \frac{1}{(2\pi\sigma h_t)^{\frac{1}{2}}} \int_0^\infty \omega_t^{-\frac{(v+1)}{2}-1} exp\left\{-\left(\frac{u_t^2}{2\sigma h_t}+\frac{v}{2}\right)\frac{1}{\omega_t}\right\}d\omega_t$$

$$\pi(\Delta \mid u) = \left(\tfrac{v}{2}\right)^{\frac{Tv}{2}}\left[\tau\left(\tfrac{v}{2}\right)\right]^{-T} \prod_{t=1}^{T} \frac{1}{(2\pi\sigma h_t)^{\frac{1}{2}}} \Gamma\left(\frac{v+1}{2}\right)\left[\frac{u_t^2}{2\sigma h_t}+\frac{v}{2}\right]^{-\frac{(v+1)}{2}}$$

$$\pi(\Delta \mid u) = \left[\frac{\Gamma\left(\frac{v+1}{2}\right)}{\tau\left(\frac{v}{2}\right)(\pi v)^{\frac{1}{2}}}\right]^T \prod_{t=1}^{T} \frac{1}{(\sigma h_t)^{\frac{1}{2}}}\left[\frac{u_t^2}{v\sigma h_t}+1\right]^{-\frac{(v+1)}{2}}$$

which is the likelihood function of the parameters Δ=(α,β,v) where

$$u_t \sim iidS_v\left(0, \frac{v-2}{v}h_t\right)$$

$$h_t = \alpha_0 + \sum_{i=1}^{p}\left(\alpha_i I_{\{u_{t-i}\geq 0\}} + \alpha_i^* I_{\{u_{t-i}<0\}}\right)u_{t-1}^2 + \sum_{j=1}^{q}\beta_j h_{t-j}$$

Proposition 2. Let a T x 1 vector y = $(y_1, y_2, \dots, y_T)'$ and a T x m matrix **X**, then we can express the approximate likelihood function of γ as follows:

$$L(\gamma \mid y, X) \propto exp\left\{-\frac{1}{2}(y - X'\gamma)'\Sigma^{-1}(y - X\gamma)\right\}$$

And the posterior density of γ as

$$p(\gamma \mid \mu_\gamma, \Sigma_\gamma) \propto exp\left\{-\frac{1}{2}(\gamma-\mu_\gamma)'\Sigma_\gamma^{-1}(\gamma-\mu_\gamma)\right\}.$$

Then the posterior density of γ is given by

$$\pi(\gamma \mid y, X) \alpha\ exp\left\{-\frac{1}{2}\left[(\gamma - A^{-1}B)'A(\gamma - A^{-1}B)\right]\right\} \tag{5}$$

3.0 Theoretical Results

Deriving the Posterior Density of the GJR-GARCH Parameters

To derive the posterior densities of the parameters α and β, let us first transform the conditional variance by defining

$$l_t = v_t - h_t \Rightarrow h_t = v_t - l_t,$$

where $v_t = u_t^2$, then

$$v_t = \alpha_0 + \sum_{i=1}^{q}\left(\alpha_i I_{\{u_{t-i}\geq 0\}} + \alpha_i^* I_{\{u_{t-i}<0\}}\right)u_{t-1}^2 + \sum_{j=1}^{p}\beta_j v_{t-j} - \sum_{j-1}^{p}\beta_j l_{t-j} + l_t.$$

We expressed l_t as

$$l_t = (x_1^2 - 1)h_t,$$

χ_1^2 denotes a Chi-squared variable with one degrees of freedom with mean equal to 1 and variance equal to 2.

The auxiliary model can be written as

$$z_t(\alpha,\beta) = v_t - \alpha_0 - \sum_{i=1}^{q}\left(\alpha_i I_{\{u_{t-i}\geq 0\}} + \alpha_i^* I_{\{u_{t-i}<0\}}\right)u_{t-1}^2 - \sum_{j=1}^{p}\beta_j v_{t-j} + \sum_{j=1}^{p}\beta_j z_{t-j}(\alpha,\beta) \tag{6}$$

where z_t is a function of (α, β) and $z_0 = v_0 = 0$. The approximate likelihood function of (α,β) is

$$L(\alpha, \beta \mid \gamma, y, X) \propto (det\ \Lambda)^{\frac{1}{2}} exp\left\{-\frac{1}{2}z'\Lambda^{-1}z\right\},$$

with $z = (z_1, z_2, \dots, z_T)'$ and $\Lambda = diag\left(\{2h_t^2\}T_{t=1}\right)$

Let us define the following recursive transformation

$$w_t^* = 1 + \sum_{j=1}^{p}\beta_j \omega_{t-j}^*$$

$$v_t^* = \sum_{i=1}^{q} v_{t-i} I_{\{u_{t-i}\geq 0\}} + \sum_{j=1}^{p}\beta_j v_{t-j}^*$$

$$v_t^{**} = \sum_{i=1}^{q} v_{t-i} I_{\{u_{t-i}<0\}} + \sum_{j=1}^{p}\beta_j v_{t-j}^{**}, \tag{7}$$

where $w_t^* = v_t^* = v_t^{**} = 0\ for\ t \leq 0$

Proposition 3. Let the (2q+1) x 1 vector c_t be given by

$$c_t = (W_t^*, v_{t-1}^*, \dots, v_{t-q}^*, v_{t-1}^{**}, \dots, v_{t-q}^{**})'$$

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_q, \alpha_1^*, \dots, \alpha_q^*)' \tag{8}$$

Then the expression (6) can be written as

$$z_t = v_t - c_t'\alpha.$$

That is, the function z_t in equation (6) can be expressed as a linear function of vector α.

Proof:

For the proof of this theorem, we use mathematical induction.

For t=1,

$$v_1 - c_1{}'\alpha = v_1 - \alpha_0 = z_1$$

Therefore, it is true for t = 1.
Let us assume that it is also true for t = k.
For t=k+1,

$$v_{k+1} - c_{k+1}{}'\alpha = v_{k+1} - \left(\alpha_0 w_{k+1}^{*} + \alpha_1 v_k^{*} + \ldots + \alpha_p v_{k+1-q}^{*}\alpha_1^{*} v_k^{**} + \ldots + \alpha_p^{*} v_{k+1-q}^{**}\right)$$

Now from the recursive transformation,

$$v_{k+1} - c_{k+1}{}'\alpha = v_{k+1}$$
$$-[\alpha_0(1+\sum_{j=1}^{p}\beta_j w_{k+1-j}^{*}) + \alpha_1(\sum_{i=1}^{q} v_{k+1-i} I_{\{u_{k+1-i}\geq 0\}} + \sum_{j=1}^{p}\beta_j v_{k+1-j}^{*}) + \ldots$$
$$+\alpha_p(\sum_{i=1}^{q} v_{k+1-p-i} I_{\{u_{k+1-p-i}\geq 0\}} + \sum_{j=1}^{p}\beta_j v_{k+1-p-j^{*}})$$
$$+\alpha_1^{*}(\sum_{i=1}^{q} v_{k+1-i} I_{\{u_{k+1-i}<0\}} + \sum_{j=1}^{p}\beta_j v_{k+1-j}^{**}) + \ldots$$
$$+\alpha_p^{*}(\sum_{i=1}^{q} v_{k+1-p-i} I_{\{u_{k+1-p-i}<0\}} + \sum_{j=1}^{p}\beta_j v_{k+1-p-j^{**}})]$$
$$v_{k+1} - c_{k+1}{}'\alpha = z_{k+1.}$$

Therefore, it is true for $t = k + 1$.

Deriving the Posterior Density of α

Now the approxiamate likelihood function of α is

$$L(\alpha \mid \gamma, \beta, y, X) \propto exp\left\{-\frac{1}{2}(v - C'\alpha)'\Lambda^{-1}(v - C'\alpha)\right\}$$

The prior distribution of α is

$$p(\alpha \mid \mu_\alpha, \Sigma_\alpha) \propto\ exp\left\{-\frac{1}{2}(\alpha-\mu_\alpha)'\Sigma_\alpha^{-1}(\alpha-\mu_\alpha)\right\}$$

Following proposition 2, then the posterior density of α is given by

$$\pi(\alpha \mid y) \propto exp\left\{-\frac{1}{2}(\alpha-\hat{\mu}_\alpha)'\hat{\Sigma}_\alpha^{-1}(\alpha-\hat{\mu}_\alpha)\right\}$$

Deriving the Posterior Density of β

To derive the posterior density of β, let us first express the function $z_t(\alpha, \beta)$ in equation (6) as a linear function of vector β using the first-order Taylor expansion at point $\tilde{\beta}$,

$$Z_t(\beta) \approx Z_t(\tilde{\beta}) + \psi_t(\beta - \tilde{\beta}), \quad (9)$$

where $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_p)$ is the previous draw of parameter β in the M-H sampler, $\psi_t = (\psi_{t1}, \psi_{t2}, \ldots, \psi_{tp})$ is the first order derivative of Z(β) evaluate at point $\tilde{\beta}$, and

$$\psi_{ti} = -v_{t-i} + Z_{t-i}(\tilde{\beta}) + \sum_{j=1}^{p}\tilde{\beta}_j\psi_{ti-j} \quad \text{for } i = 1, \ldots, p,$$

with $\psi_{ti} = 0$ for $t \leq 0$.
Furthermore, define

$$b_t = Z_t(\tilde{\beta}) + \varphi_t\tilde{\beta} \qquad where\ \varphi_t = -\psi_t.$$

Then, from the equation (9), we have $Z_t(\beta) \approx b_t - \varphi_t\beta$.

Moreover we define $b = (b_1, \ldots, b_T)$ and T x p matrix φ, hence we can approximate ***Z*** as

$$Z \approx b - \varphi\beta.$$

The approximate likelihood function of parameter β as,

$$L(\beta \mid \alpha, \gamma, y, X) \propto exp\left\{-\frac{1}{2}(b-\varphi\beta)'\Lambda^{-1}(b-\varphi\beta)\right\}.$$

The prior density of β is given by

$$p = (\beta \mid \mu_\beta, \Sigma_\beta) \propto exp\left\{-\frac{1}{2}(\beta-\mu_\beta)'\Sigma_\beta^{-1}(\beta-\mu_\beta)\right\}$$

By proposition 2, the posterior density of βis the combination of likelihood function and the prior density, that is,

$$\pi(\beta \mid \alpha, \gamma, y, X) \propto exp\left\{-\frac{1}{2}(\beta-\hat{\mu}_\beta)'\hat{\Sigma}_\beta^{-1}(\beta-\hat{\mu}\beta)\right\}$$

Deriving the Posterior Density of ω

The full conditional density of ω is straightforward to derive. Let $\omega_1, \omega_2, \ldots, \omega_T$ are independent and identically distributed random variables from an Inverted Gamma density given by,

$$p(\omega_t \mid v) \propto \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\tau\left(\frac{v}{2}\right)}\omega_t^{-\frac{v}{2}-1}\ exp\left\{-\frac{v}{2\omega_t}\right\}$$
$$\propto \omega_t^{-\frac{v}{2}-1}\ exp\left\{-\frac{v}{2\omega_t}\right\}.$$

Then the joint density of T x 1 vector $\omega = (\omega_1, \omega_2, \ldots, \omega_T)$ is

$$p(\omega \mid v) \propto \prod_{t=1}^{T}\omega_t^{-\frac{v}{2}-1}\ exp\left\{-\frac{v}{2\omega_t}\right\}.$$

Then the likelihood function of ω is

$$L(\omega \mid y, X) \propto \prod_{t=1}^{T}(\omega_t)^{-\frac{1}{2}}\ exp\left\{-\frac{1}{2}\left[\frac{(y_t - x_t'\gamma)^2}{\omega_t h_t}\right]\right\}.$$

Then using Bayes Theorem, we obtain the joint posterior as

$$\pi(\omega \mid u) \propto L(\omega \mid u)p(\omega \mid v)$$
$$\propto \prod_{t=1}^{T}(\omega_t)^{-\frac{v+1}{2}-1}\ exp\left\{-\frac{1}{2\omega_t}\left[\frac{(y_t - x_t'\gamma)^2}{\sigma h_t} + v\right]\right\},$$

which is the kernel of an Inverted Gamma Density with parameters

$$\frac{v+1}{2} \text{ and } \frac{1}{2}\left[\frac{(y_t - x_t'\gamma)^2}{\sigma h_t} + v\right].$$

Deriving the Posterior Density of *v*.

The translated exponential density with parameters $\lambda > 0$ and $\delta \geq 2$ is given by

$$p(v) = \lambda\ exp[-\lambda(v-\delta)].$$

The prior density of vector $\omega = (\omega_1, \omega_2, \ldots, \omega_T)$ conditional on *v* given in (4) is

$$p(\omega \mid v) = \left(\frac{v}{2}\right)^{\frac{Tv}{2}}\left[\tau\left(\frac{v}{2}\right)\right]^{-T}\prod_{t=1}^{T}\omega_t^{-\frac{v}{2}-1}\ exp\left\{-\frac{v}{2\omega_t}\right\}.$$

Thus, the posterior density is

$$p(v \mid \omega) \propto p(\omega \mid v).p(v)$$

$$\left(v \middle| \omega\right) \propto \left(\frac{v}{2}\right)^{\frac{Tv}{2}} \left[\tau\left(\frac{v}{2}\right)\right]^{-T} exp\left\{-\Delta v\right\}$$

$$\text{where } \Delta = \lambda + \frac{1}{2}\left[\sum_{t=1}^{T}\left(\frac{1}{\omega_t} + \ln ln\omega t\right)\right].$$

With the use of these derived posterior densities of the parameters α, β, ω, and v in the GJ-GARCH model, it can be surmised that these posterior densities are essential for Bayesian inference and parameter estimation.

4.0 Conclusion

The primary aim of this study is to provide estimates of the GJR-GARCH (p, q) model with student-t error distribution using the Bayesian approach. For the model with student-t distribution, the likelihood function on u_t, as shown in equation (1), provide difficulties in the Bayesian framework. Thus, the model is expressed as an equivalent to the model in equation (3), where there is an additional parameter ω_t, assumed to be an Inverted Gamma with parameters $\frac{v}{2}$ and $\frac{v}{2}$. The derivation of the posterior densities of the GJR-GARCH parameters α and β were derived in this paper. Also, the latent variable ω_t is directly sampled from the full conditional density since the distribution of ω_t is already available.

ACKNOWLEDGEMENT

Sincere appreciation and heartfelt gratitude for the help and support are extended to the following people for their important contribution in making this study possible. To DOST-SEI Accelerated Science and Technology Human Resource Development Program for granting the author full scholarship and financial support to finish her graduate academic pursuit and research study, and to Dr. Arnulfo P. Supe, her research adviser, for his never-failing guidance, passion to impart his knowledge, skills, comments, suggestion and motivation in completing this piece of work.

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