HOGATT-HANSELL-TYPE IDENTITIES OF ODD NUMBERS

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Abstract

This paper is a kind of investigatory project that can be modeled by students in the basic education doing mathematical research. The results established here may be incorporated in the instructional materials for teaching function and sequence. The number o(n,k) for positive integers n and k from the pyramid of odd numbers φ_n is introduced and some Hogatt-Hansell-type identities are established and discussed.

Keywords: *pyramid of odd numbers, explicit formula, Hogatt-Hansell-type identity*

1.0 Introduction

Over the years, Pascal's triangle is one of the well-known pyramids of numbers. It is an efficient technique in generating the binomial coefficients $\binom{n}{k}$ and some of its interesting properties (Koshy, 2007).

One of which is the Hogatt-Hansel identity mathematically expressed as:

$$\binom{n-1}{r-1}\binom{n}{r+1}\binom{n}{r+1} = \binom{n-1}{r}\binom{n+1}{r+1}\binom{n}{r-1}$$

which is obtained from the alternate vertices of a regular hexagon formed from the binomial coefficients in any three adjacent rows in the Pascal's triangle (Hoggatt & Hansell, 1971). This identity also implies that the product of all the numbers at the vertices of the hexagon is a square. Subsequent works have generalized this triangular array to a tableau and called it as *generalized hidden hexagon squares* (Stanton & Cowan, 1970; Gupta, 1974). They established that the product of the six binomial coefficients spaced around $\binom{m}{n}$ is a perfect integer. Thus, this present will deal with some Hogatt-Hansell-type identities from the pyramid of odd numbers φ_n generated by the number *o(n,k)* for positive integers *n* and *k*.

In the same year, Hogatt and Bicknell (1974) published some properties of triangular numbers which exposed Fibonacci's triangle of odd numbers. It features the arithmetic triangle of odd numbers in which the n^{th} row has n entries, the center element is n^2 for even n and the row sum is n. They confirmed that every odd number is the difference of two consecutive squares and that the difference of the squares of two consecutive triangular numbers is a perfect cube. Moreover, they have established the generating function of the cube number generated from the triangle. Parallel to this work, Conway & Guy (1996) found out that the triangle numbers T_n can be related to the square numbers by: $(2n + 1)^{2} = 8T_n + 1 = T_{n-1} + 6T_n + T_{n+1}$ and established the generating functions of T_n.

Motivated by the above-cited works, this paper will exploit the pyramid of odd numbers and discuss the explicit form of its elements to establish some Hogatt-Hansell-type identities. The visual geometric patterns of the number $o(n,k)$ on the φ_n is envisioned to extend its applications on functions, sequences, and in the field of engineering. Moreover, the concepts can be integrated in the instructional materials for teaching mathematics; thus, helping the students develop higher order thinking

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skills.

2.0 Results & Discussions

Consider the sequence of odd numbers $X_n = \{1, 3, 5, 7, 9, \ldots, 2n\text{-}1\}$. The elements of X_n for $n = 7$ are arranged in a triangular array as shown in fig. 1.

Let φ_n be the pyramid of odd numbers with n rows and k columns. Every element in the φ_n is denoted by $o(n,k)$, which is explicitly defined by $o(n,k)=n(n-1)+2k-1$ for $n \geq k \geq 1$. Observe first few elements of φ_n in fig. 1 as follow: $o(1,1) = 1$, $o(3,2) = 3\ (3\text{-}1) + 2\ (2)\ \text{-}1 = 9$ and $o(5,3) = 5(4) + 2(3) - 1 = 25$. Moreover, $o(n,1) = n\ (n-1) + 1$ and $o(n,n) = o(k,k)$, for $n = k$. Note that $o(n,k)$ is always an odd number for every positive integers n and k. Hence, the k^{th} element in the n^{th} row of φ_n is given as $o(n,k) = n\ (n-1) + 2k - 1$. For $n = k$, $o(n,n) = o(k,k)$. Note further that in fig. 1, it shows that for n=k, the last element is $o(k,k)$. Hence, $o(n,n)$ or $o(k,k)$ denotes the last element in the k^{th} row of φ_n. This remark is essential in establishing some properties of φ_n.

In the subsequent theorems, the numbers $o(n,k)$ will be interpreted in different forms of equalities using square and pentagon sub-arrays of φ_n. Number equality is the highlight of this identity for all square sub-arrays of four adjacent elements in φ_n for $n \geq 3$. The following figures will help us to better understand the identities with sub-arrays of φ_n. From fig. 2, we can form the first square sub-array in the 2nd and 3rd rows of φ_3. Note that the sums of elements in the diagonals are equal; that is 3+9 = 5+7. On the other hand, we can form two square sub-arrays in the 3rd and 4th rows of φ_4 in addition to the square sub-array in the 2nd and 3rd rows of φ_4. Note further that those diagonals of

$n \backslash k$							
1	1						
2	3	5					
3	7	9	11				
4	13	15	17	19			
5	21	23	25	27	29		
6	31	33	35	37	39	41	
7	43	45	47	49	51	53	55

Figure 1. The φ_7 and its elements $o(n,k)$

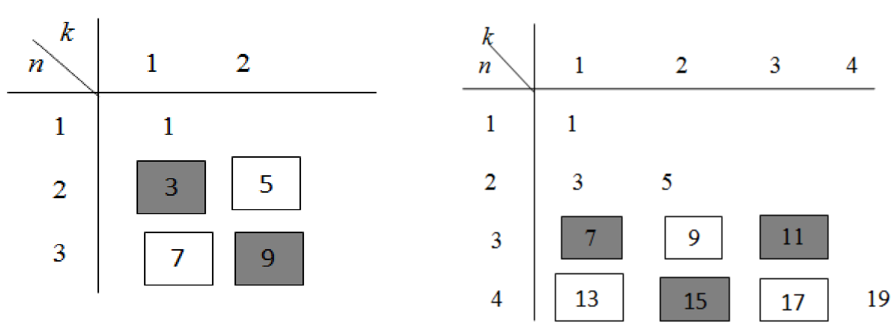


Figure 2. Square sub-array in φ_n for $n = 3, 4$

$n \backslash k$	k	$k+1$
n	$o(n,k)$	$o(n,k+1)$
$n+1$	$o(n+1,k)$	$o(n+1,k+1)$

Figure 3. Generalized form of square sub-array

each square show equal sums; that is, *7+15=9+13* for the left-side array and *9+17=11+15* for the right-side array. Taking an arbitrary n and k in the φ_n, as shown in fig. 3, we can generalize this relationship as follows: $o(n,k)+o(n+1,k+1)= o(n+1,k) + o(n,k+1)$. Hence, we come up with the following theorem.

Theorem 1. *The alternate sums of four adjacent elements in* φ_n *are equal. That is, for* $n \geq 3$,

$$o(n,k)+o(n+1,k+1)=o(n+1,k) + o(n,k+1).$$

Proof:

$$\begin{aligned} o(n,k)+o(n+1,k+1) &= [n(n-1)+2k-1]+[(n+1)(n+1-1)+2(k+1)-1] \\ &= n(n-1)+2k-1+(n+1)n+2k+2-1 \\ &= [n(n-1)+2k-1]+[(n+1)n+2k+1] \\ &= [(n+1)n+2k-1]+[n(n-1)+2k+1] \\ &= [(n+1)[(n+1)-1] + 2k-1] + [n(n-1)+2(k+1)-1] \\ &= o(n+1,k) + o(n,k+1) \end{aligned}$$

Thus,

$$o(n,k)+o(n+1,k+1) = o(n+1,k) + o(n,k+1) \quad \blacksquare$$

Adding the distance of the square sub-arrays will generate another interesting result. Let us observe fig. 4 on how the distance of the elements vary with adjacent elements in forming a square sub-array.

Notice that if you add the diagonal elements of fig.4(a) you will have *7+15+25 = 11+15+21*. Now, if you expand the array by adding two rows (down) and column (to the right) as shown in Figure 4(b), we have the

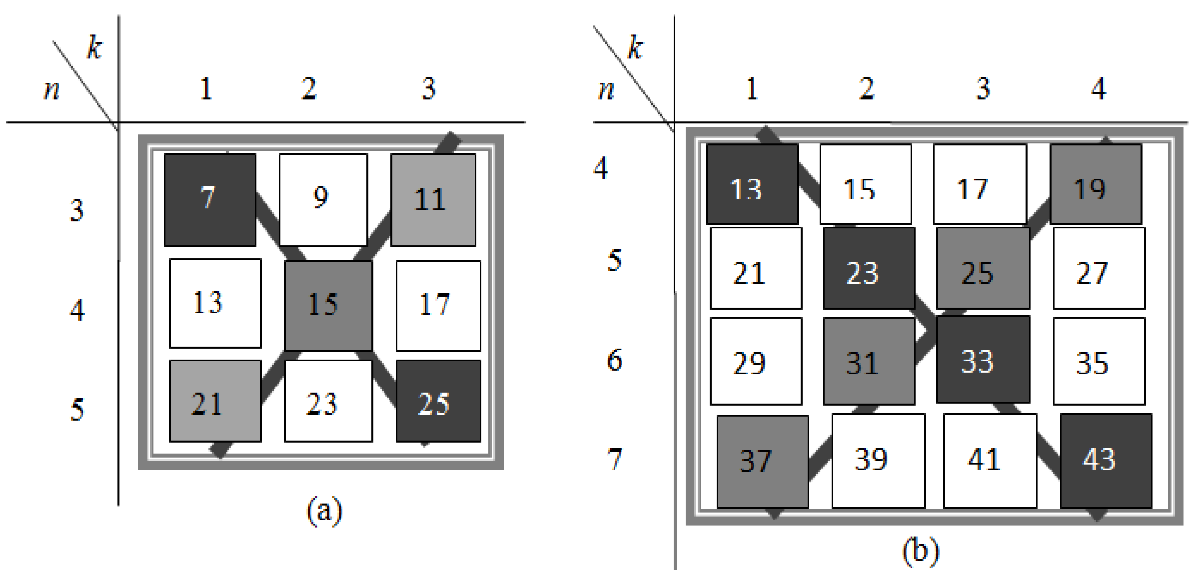


Figure 4. Sub-arrays with length m = 2, 3 in φ_n

the sum of diagonal elements such that *13+23+33+43 = 112 = 19+25+31+37.* Thus, Figure 4(b) is the consequence if the length of the square sub-arrays is increased. Hence, the next theorem generalizes this observation.

Theorem 2. *Let δ_m be a square of sub-array of φ_n with m rows $m \le n$. Then the sums of the elements in the diagonal sub-array of δ_m are equal. That is, if $o(1,j)\in\delta_m$, then,*

$$\sum_{j=0}^{m-1} o(n+j,k+j) = \sum_{j=1}^{m-1} o(n+j-1,k+m-j)$$

where $k \le n - m + 1$ and $m \le n-2$.

Proof:

For positive integers m, k and n, where $k \le n - m$ and $m \le n-2$ we have:

$$\begin{aligned}\sum_{j=1}^{m} o(n+j-1,k+m-j) &= \sum_{j=1}^{m}[(n+j-1)(n+j-2)+2(k+m-j)-1]\\ &= \sum_{j=1}^{m}(n+j-1)(n+j-2)+\sum_{j=1}^{m}[2(k+m-j)-1]\\ &=\sum_{j=0}^{m-1}(n+j)(n+j-1)+\sum_{j=0}^{m-1}[2(k+j)-1]\\ &=\sum_{j=0}^{m-1}[(n+j)(n+j-1)+2(k+j)-1]\\ &=\sum_{j=0}^{m-1} o(n+j, k+j). \quad \blacksquare\end{aligned}$$

Pascal's triangle exhibited some intriguing identities on the equality of the product of alternate vertices in any hexagon formed within its adjacent rows and proved that the product of six vertices is a square (Koshy, 2007) and even generalized by Gupta (1974).

Similarly, φ_n contains an identity on six elements equally spaced around $o(n,k)$ whose sums of three alternate elements in horizontally-oriented pentagon are equal considering the m length of space around $o(n,k)$. For m = 1, the sums of the alternate elements around $o(n,k)$ of φ_n are equal or form two consecutive odd positive integers. These identities are illustrated in fig. 5.

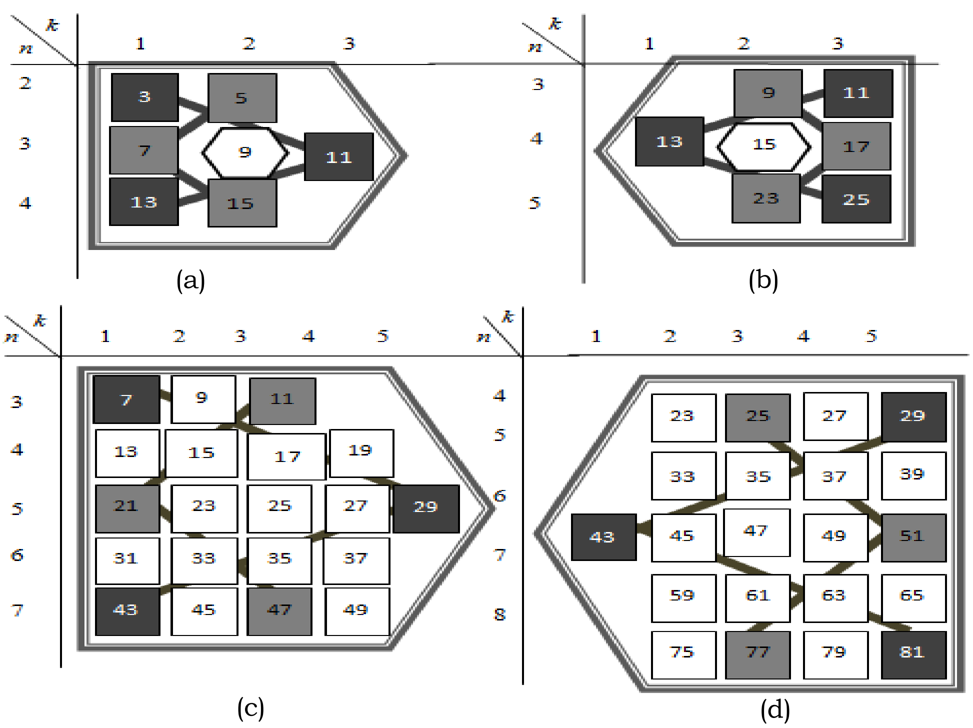


Figure 5. Horizontal Orientation of Pentagon Sub-arrays of φ_n for m = 1, 2

Observe that for $m = 1$, we have 3+11+13 = 5+7+15 (as shown in Figure 5a) and 11+13+25 = 9+17+23 (as shown in Figure 5b). For $m = 2$, alternate sums yield the following: 7+29+43 = 11+21+47 (as shown in Figure 5c) and 29+43+81 = 25+51+77 (as shown in Figure 5d). Thus, illustration show that adding the 3 alternate elements in a pentagon of opposite form in j_n with different length will always give equal sum.

However, this type of sub-array cannot be formed for $n \le 3$. The following theorem embodies the results illustrated in the preceding discussion.

Theorem 3. *For horizontally oriented pentagon sub-array of φ_n, the sums of alternate elements equally spaced around o(n,k) by length m are equal. That is,*

$$o(n\text{-}m,k\text{-}m)+o(n,k+m)+o(n+m,k\text{-}m) = o(n,k\text{-}m)+o(n\text{-}m,\ k)+o(n+m,k)$$

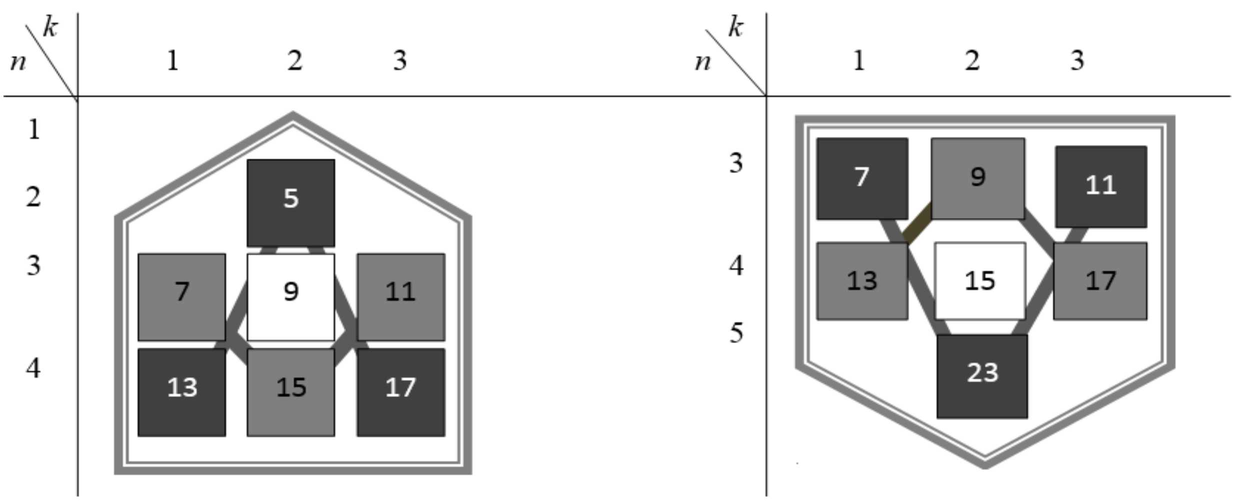
where $n \ge 3$ *and* $n \ge 4$.

Proof. The LHS of (3) becomes

$$\begin{aligned} \text{LHS} &= o(n\text{-}m)(n\text{-}m\text{-}1)+2(k\text{-}m)\text{-}1+n(n\text{-}1)+2(k+m)\text{-}1+(n+m)(n+m\text{-}1)+2(k\text{-}m)\text{-}1 \\ &= (n\text{-}m)(n\text{-}m\text{-}1)+2k\text{-}2m\text{-}1+n(n\text{-}1)+2k+2m\text{-}1+(n+m)(n+m\text{-}1)+2(k\text{-}m)\text{-}1 \\ &= (n\text{-}m)(n\text{-}m\text{-}1)+2k\text{-}1+n(n\text{-}1)+2k\text{-}1+(n+m)(n+m\text{-}1)+2(k\text{-}m)\text{-}1 \\ &= n(n\text{-}1)+2(k\text{-}m)\text{-}1+(n\text{-}m)(n\text{-}m\text{-}1)+2k\text{-}1+n+m)(n+m\text{-}1)+2k\text{-}1 \\ &= o(n,k\text{-}m)+o(n\text{-}m,k)+o(n+m,k) = \text{(RHS)} \ \square \end{aligned}$$

Taking also into consideration a vertically-oriented pentagon sub-array of φ_n for $m = 1$ shown in fig. 6, again, the alternate sums provide identity of φ_n elements around $o(n,k)$. Note that in fig. 6(a) two consecutive odd numbers are generated from the sums of alternate elements in the pentagon sub-array. That is, 5+13+17 = 35 and 7+11+15 = 33. Similarly, in fig. 6(b) gives the sums 7+23+11=41 and 9+13+17 = 39. These results show alternate sums of six elements around $o(n,k)$ of φ_n, in a pentagon sub-arrays, forming two consecutive odd positive integers. The following theorem formally stated the aforementioned results.

Theorem 4. *The alternate sums of the six elements around $o(n,k)$ of φ_n in a vertically-oriented pentagon form two consecutive odd positive integers.*



(a) (b)

Figure 6. Vertical orientation of pentagon sub-arrays for $m = 1, 2$

Proof. Let O_1 and O_2 be the alternate sums of the six elements around $o(n,k)$ of φ_n, where

$$O_1 = o(n,k-1)+o(n,k+1)+o(n+1,k)$$

and

$$O_2 = o(n-1,k)+o(n+1,k-1)+o(n+1,k+1).$$

then,

$$\begin{aligned} O_1 &= n(n-1)+2(k-1)-1+n(n-1)+2(k+1)-1+n(n+1)+2k-1 \\ &= 2n(n-1)+n(n+1)+6k-3 \\ &= n(3n-1)+3(2k)-3 \end{aligned}$$

Note that $n(3n-1)$ is even integer for $\forall n \in Z^+$ and $3(2k) - 3$ is odd for every positive integer k. Hence, $n(3n-1) + 3(2k) - 3$ odd. Thus, O_1 is odd. Similarly,

$$\begin{aligned} O_2 &= (n-1)(n-2)+2k-1+n(n+1)+2(k-1)-1+n(n+1)+2(k+1)-1 \\ &= 2n(n+1)+(n-1)(n-2)+6k-3 \\ &= n(3n-1)+3(2k)-1 \end{aligned}$$

Observe that $n(3n-1)$ is even, for any positive n and $3(2k)-1$ is odd for any positive integer k. Thus, $n(3n-1) + 3(2k)-1$ is odd. Hence, O_2 is odd. Now,

$$\begin{aligned} O_2 &= n(3n-1) + 3(2k) - 1 \\ &= [n(3n-1) + 3(2k) - 3]+2 \\ &= O_1 + 2 \end{aligned}$$

Therefore, O_1 and O_2 are consecutive positive odd integers.

3.0 Recommendations for Future Studies

This paper exploited the sum of odd numbers equally spaced around $o(n,k)$ in a generalized square sub-array, and horizontally and vertically-oriented pentagon sub-array in exposing Hogatt-Hansell-type identities. More identities of this type and extensions of properties of odd numbers may be established if the product will be investigated as with the work of Gupta (1974). On the other hand, other special numbers such as stirling-type numbers may be ventured for possible extensions parallel to that of the odd numbers in the pyramid or triangle. Finally, some combinatorial interpretations such as the 0-1 tableaux may be explored to establish more interesting combinatorial properties of $o(n,k)$ for refinement and greater applicability of this number sequence.

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