

CORRELATING TWO FRACTAL RANDOM VARIABLES: APPLICATION TO ANIMAL FOOD SEARCH BEHAVIOR

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ABSTRACT

The paper proposed two (2) measures for correlating fractal random variables. The correlation measures depicted the extent to which roughness in one variable induces roughness in the other. The first measure, fractal correlation, directly used the usual moment-based product moment correlation of the fractal measures λ_x and λ_y at each point (x_i, y_i) . Each of the fractal measures was marginally exponentially distributed and so their second moments exist. The second measure, fractal geometric correlation, took advantage of the fact that fractal curves may have infinite perimeters but they enclosed finite areas A . The fractal geometric correlation was given by the reciprocal of $(A+1)^{\lambda-1}$. Simulation results were given and applications in landscape ecology were provided.

Keywords: fractal geometric correlation, fractal correlation measure, statistical fractals

1.0 Introduction

In earlier papers of Padua et al., 2012; 2013, a univariate fractal probability model was developed which included, among others, a measure of roughness of the fractal random variable X through the fractal dimension (λ) . The model uses the simple power law model:

$$(1) f(x; \lambda; \theta) = \frac{\lambda - 1}{\theta} \left(\frac{x}{\theta}\right)^{-\lambda}, x \geq \theta, \lambda > 1, \theta > 0$$

For most applications, the fractal dimension λ lies between 1 and 2, $1 < \lambda < 2$, and so the first and second moments of x do not exist. The maximum likelihood estimator of λ based on model 1 is:

$$(2) \hat{\lambda} = 1 + n \left(\sum_{i=1}^n \log \left(\frac{x_i}{\theta} \right) \right)^{-1}$$

Thus, for $n=1$ each observation x_i has an associated roughness $\hat{\lambda}_i$ which takes the place of a derivative at x_i in the usual

(continuous) model. The distribution of these roughness measures was shown to obey an exponential distribution. Consider two fractal random variables X and Y with associated joint fractal distribution $f(x, y)$. We propose to model the joint distribution as:

$$(3) f(x, y) = A \left(\frac{x+y}{\theta} \right)^{-\lambda}, x \geq \theta_1, y \geq \theta_2, \lambda > 1, \theta = 0,$$

else,

Here, $\theta = \max \{\theta_1, \theta_2\}$ and attempt to define a correlation measure between x and y . The obvious Pearson “r” measure is inappropriate in this case because the first two moments of (3) may not exist at all. It follows that a correlation measure appropriate for model (3) should not have any reference to the existence of a second moment.

There are various conceptual difficulties that need to be addressed

along this subject. First, a “correlation measure” for fractal random variable measures something more than the usual Pearson “r” correlation coefficient. It is not concerned so much with the question of whether an increase (decrease) in x corresponds to an increase (decrease) in y but whether x “causes” the observed roughness in y . Second, because the notion of “roughness” is a relative concept, any fractal correlation coefficient is also necessarily relative. Thus, an observed causal relationship between x and y in one context may not hold true in another context. For instance, poverty (x) may induce a corresponding “roughness” on the incidence of crimes (y) in one municipality but need not be true in another municipality. Correlation measures derived from the concept of “mutual information” are generally of this nature.

2.0 Methodology (Proposed Correlation Measures)

2.1 Mutual Information

A classical measure used to determine how much information on variable x provides information on y is the mutual information index:

$$(4) MI = \iint f(x, y) \log \frac{f(x, y)}{f(x)f(y)} dx dy.$$

Note that if x and y are independent random variables, then $MI = 0$. There are serious implementation problems associated with the use of (4), the most troublesome being that of estimating the joint density $f(x, y)$. We therefore prefer not to use this measure.

2.2 Fractal Dimension Correlation

Alternatively, we can use the roughness of x and y as indicators on the

way roughness in one variable induces roughness in the other as follows: Let:

$$(5) \hat{\lambda}_x(x_i) = 1 + \frac{1}{\log(\frac{x_i}{\hat{x}_1})}, \quad \hat{\lambda}_y(y_i) = 1 + \frac{1}{\log(\frac{y_i}{\hat{y}_1})}, \quad i = 1, 2, \dots, n$$

be the maximum likelihood estimators of λ_x and λ_y for a single observation. The pairs $\{\hat{\lambda}_x(x_i), \hat{\lambda}_y(y_i)\}$ are then correlated:

$$(6) r_{\lambda_x \lambda_y} = \frac{\sum_{i=1}^n (\hat{\lambda}_x(x_i) - \bar{\lambda}_x)(\hat{\lambda}_y(y_i) - \bar{\lambda}_y)}{\sqrt{\text{var}(\hat{\lambda}_x)} \sqrt{\text{var}(\hat{\lambda}_y)}}$$

We take (6) as an estimate of the theoretical correlation

$$(7) \rho_{\lambda_x \lambda_y} = \frac{\text{cov}(\lambda_x, \lambda_y)}{\sqrt{\text{var} \lambda_x} \sqrt{\text{var} \lambda_y}}$$

The existence of (7) is guaranteed by the fact that the distribution of the fractal dimensions λ_x and λ_y are each marginally distributed exponentially:

$$(8) f(\lambda) = a e^{-a\lambda}, \quad \lambda > 0.$$

Marshall and Olkin (1967) provided the bivariate exponential distribution in the context of reliability of a two component system subjected to three types of Poisson shocks a_1 , a_2 and a_3 caused failure in the first, second and both components respectively (Malik and Trudel, 1986). The bivariate Marshall—Olkin exponential distribution is:

$$(9) F(\lambda_x, \lambda_y) = 1 - \exp(-a_1 \lambda_x - (a_2 + a_3) \lambda_y) + \exp(-a_1 \lambda_x - a_2 \lambda_y - a_3 \max\{\lambda_x, \lambda_y\})$$

where, $a_1, a_2, a_3 > 0$ and $\lambda_x, \lambda_y > 0$. Equivalently,

$$(10) H(\lambda_x, \lambda_y) = P(\lambda_x > \lambda_x, \lambda_y > \lambda_y) = \exp(-a_1 \lambda_x - a_2 \lambda_y - a_3 \max\{\lambda_x, \lambda_y\})$$

Note that $E(\lambda_x) = \frac{1}{\alpha_1}$, $E(\lambda_y) = \frac{1}{\alpha_2}$ so that we only need to find $E(\lambda_x, \lambda_y)$ or the expectation of the product of two exponential random variables. Let $\mu = \lambda_x$,

λ_y , then:

$$(11) H(\mu) = \int_{\sqrt{\mu}}^{\infty} \frac{1}{v} \exp\left(-\frac{\alpha_2 \mu}{v} - (\alpha_2 + \alpha_3)v\right) dv$$

where $v = \lambda_y$, $a_1, a_2, a_3, v > 0$. Upon noting that:

$$(12) F(\lambda_x, \lambda_y) = H(x, y) + F(x) + F(y) - I,$$

we can obtain $F(\mu)$ as:

(13) $F(\mu) = 1 - H(\mu)$ (Malik and Trudel, 1986). The expected value of $\mu = \lambda_x, \lambda_y$ is therefore,

$$(14) E(\mu) = \int_0^{\infty} \mu dF(\mu) < \infty$$

which is easily seen to be finite. Numerical integration is needed to find (11) and (14).

Lewis et al. (1981) found a more practical way to obtain the correlation between two exponential random variables. For positively dependent exponential model, let E_1 and E_2 be independent and identically distributed (iid) exponential random variables with unit mean.

Define:

$$(15) X_1 = \beta E_1 + I E_2$$

$$X_2 = \beta E_2 + I E_1$$

where I is a Bernoulli random variable with $P(I = 0) = \beta$ and $P(I = 1) = 1 - \beta$. Then, X_1 has an exponential distribution because the moment—generating function of X_1 is:

$$(16) \varphi_{x_1}(t) = \frac{1}{1+\beta t} \left\{ \beta + (1-\beta) \frac{1}{1+t} \right\} = \frac{1}{1+t}$$

Similarly, X_2 is exponential with:

$$(17) \text{corr}(x_1, x_2) = \rho_{x_1, x_2} = 3\beta(1-\beta).$$

It follows that $0 \leq \rho_{x_1, x_2} = 0.75$ where the maximum is attained at $\beta = 0.05$. The Pearson product moment correlation measures linear dependency taking

$$(18) f(x_1, x_2) = \frac{1}{1+\beta} \exp\left(-\frac{(x_1+x_2)}{1+\beta}\right) \cdot I\left(\beta x_2 < x_1 < \frac{1}{\beta} x_2\right) + \frac{1}{\beta} \exp\left(-\frac{(x_1+x_2)}{\beta}\right).$$

maximal values at ± 1 if the variables are linearly related: this is not the case here.

The joint distribution of X_1 and X_2 is: where $0 \leq \beta \leq 1, x_1, x_2 > 0$. Figure 1 shows the isometric plot of (18).

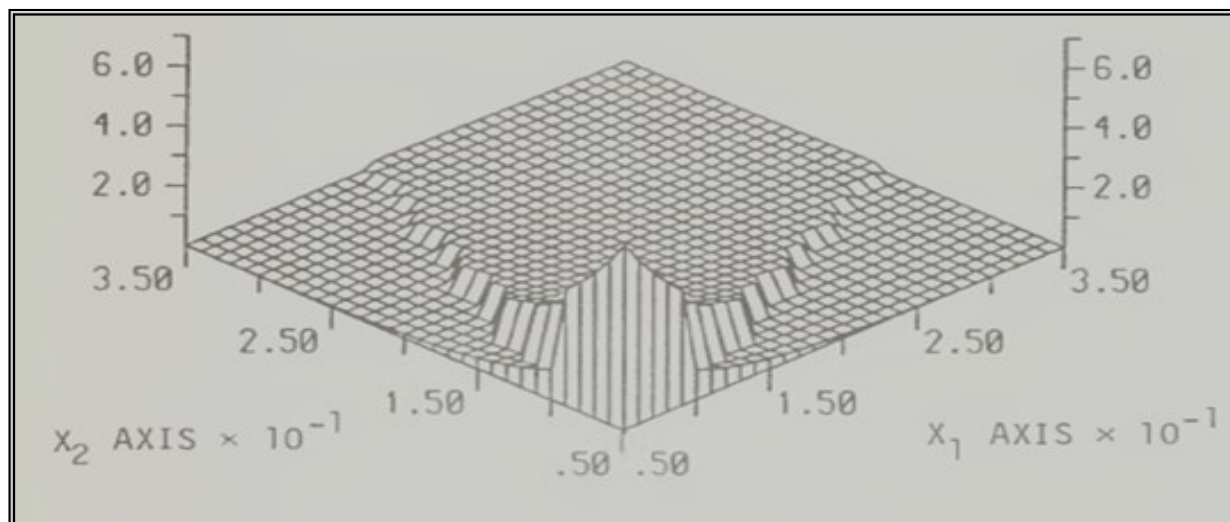


Figure 1. Isometric plot of the joint probability density function of the positively dependent EP (β) model for $\beta = 0.5$. This value of β gives the maximum attainable correlation of 0.75; the figure has been truncated for both variables at 0.5 (Source: Lawrance and Lewis, 1981, Naval Postgraduate School, Monterrey, California)

The figure shows that the joint pdf is a wedge-shaped sector of high density at $(\beta x_2 < x_1 < \frac{1}{\beta} x_2)$ super-imposed over an independent bivariate exponential diversity.

2.3 Fractal Geometric Correlation (FGC)

A more intuitive and geometric measure of association between two (2) fractal random variables can be derived that uses the original fractal geometry of the scatterplot (x_i, y_i) , $i = 1, 2, \dots, n$. Let $H = \{(x_i, y_i) | (x_i, y_i) \text{ iid } F(x, y), \text{ a bivariate fractal distribution}\}$. The scatterplot of H will be denoted by $G(H)$, the graph of H . We connect the points (x_i, y_i) and (x_j, y_j) $i \neq j$ by short line segments (called edges) in such a way that no two edges intersect except at their endpoints. The resulting closed curve is a fractal curve: $\gamma: \mathbb{R}^2 \rightarrow \mathbb{R}$ which is continuous but nowhere differentiable (non-rectifiable). This means that the perimeter P of the curve tends to infinity. However, the area A enclosed by the curve is finite and is well-defined.

Techniques for calculating areas enclosed by non-rectifiable curves are available in the literature (Sasaki, Shihaba & Hatanaka, 1994). We propose a correlation measure based on this observation.

Let $G(H)$ define a non-rectifiable curve $\gamma(\cdot)$. Let A be the area enclosed by $\gamma(\cdot)$, then

$$(19) R^\lambda = \frac{1}{(A+1)^{\lambda-1}}$$

is a measure of the roughness of Y induced by the roughness of X . Note that if $Y_i = aX_i$, $a > 0$, then $R^2 = 1$ since $A = 0$ and the fractal dimension of the straight line is $\lambda = 1$. That is, if X were a fractal random variable, then Y will also be fractal of the same dimension as X_j . The resulting figure is one that is not a fractal object (a straight line).

If $A \rightarrow \infty$, then $R^2 \rightarrow 0$, then the geometric figure whose boundaries are the observations (x, y) departs from a straight line. The fractal dimension λ of the figure increases indicating a higher degree of roughness. In fact, it is easy to see that the higher the fractal dimension of $G(H)$, the higher becomes the area A , viz, the farther are the points from the usual regression line. Hence, the measure (19) is a measure of departure from a linear relationship between x and y . The computation of the area A enclosed by the points (x, y) can be difficult when the number of points is large. The object can be so rugged as to render the computation of the area almost intractable.

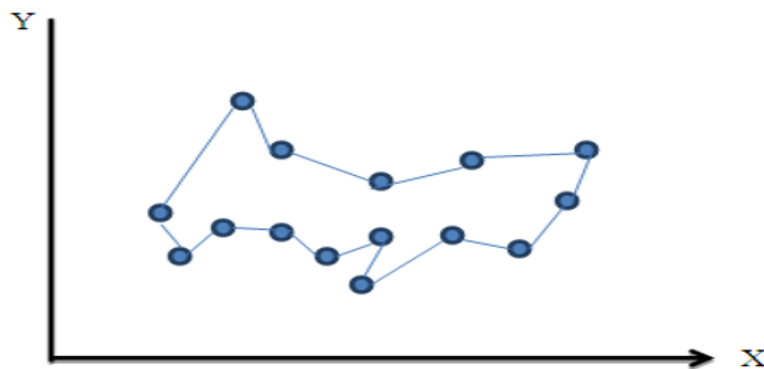


Figure 2. Non-rectifiable curve enclosing a finite area

For this reason, we propose to replace (19) by a monotone measure:

$$(20) R^\lambda = 1 - (\lambda - 1)^{Har(\lambda_x, \lambda_y)}$$

where $Har(\lambda_x, \lambda_y)$ is the harmonic mean of the fractal dimensions of x and y , respectively. Note that if the figure is a straight line $\lambda = 1$ and the resulting correlation measure is $R^\lambda = 1$. As $\lambda \rightarrow 2$, the curve traced by the points (x, y) becomes rougher and $R^\lambda \rightarrow 0$.

The usual Pearson product moment correlation coefficient is a measure of linear relationship. If the variables x and y are linearly related, then the correlation coefficient is 1 (or -1). However, even if x and y are related such as when $y = 2/x$, the Pearson correlation coefficient will be close to zero. On the other hand, the fractal geometric correlation provides a value that is not close to zero and dependent only on the fractal dimension of the curve. Furthermore, while the Pearson correlation coefficient is quite sensitive to “outliers”, the fractal geometric correlation is not.

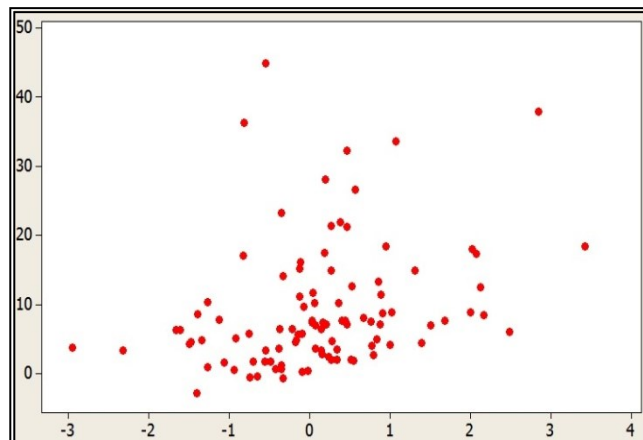
Fractal softwares can be used to calculate λ , λ_x , and λ_y (see for example FRAKOUT.COM). Below is an actual illustration of the use of Frakout software

to determine how an animal's search range for food (x) induces ruggedness on the amount of food (y) actually found by the animal.

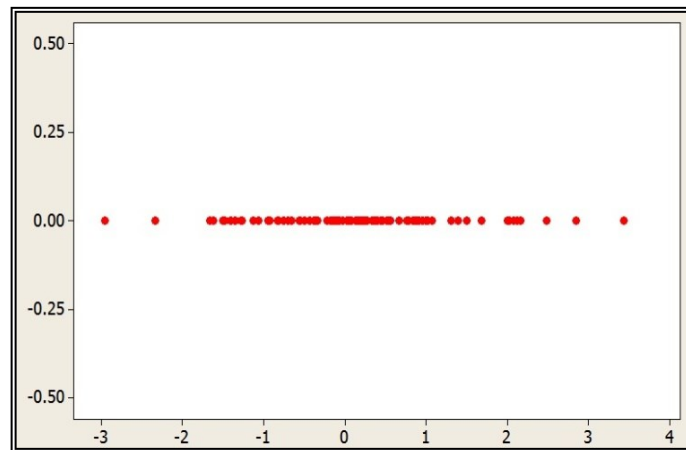
Animal Food Search Range

1. Scatterplot of variables x (range of search) and y (food).
2. Project to x -axis and show ruggedness of x (search range). (set $y = 0$)
3. Project to y -axis and show ruggedness of y (amount of food). (set $x = 0$)
4. FRACTOGRAM: Combine x - y and scatterplot
5. Open FrakOut.exe. Click File, New and Load the .jpg file.
6. Click Sheet, Add Sheet and Decrease cell size.
7. Connect and trace the points by highlighting the corresponding cells.
8. Go to EXCEL, copy the cell size and # of cells. Compute for the log (cell size). Compute for the log (# of cells).
9. Perform a linear regression analysis. The slope will give us the “fractal dimension” of the highlighted cells (ruggedness of x)
10. Connect and trace the points by highlighting the corresponding cell.

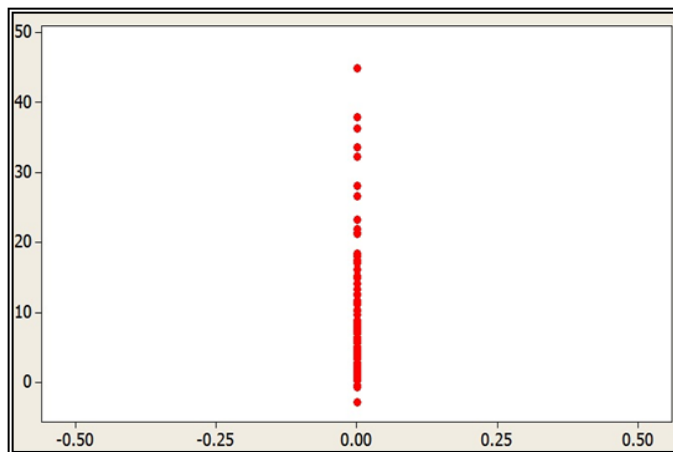
1. Scatterplot of variables x (range of search) and y (food).



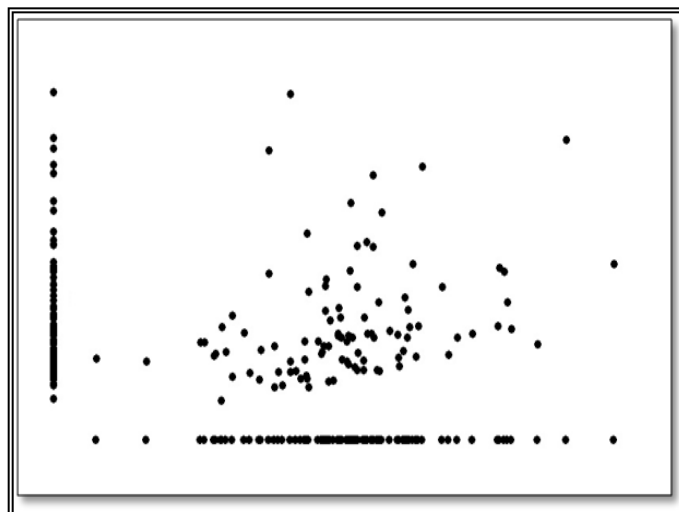
2. Project to x-axis and show ruggedness of x (search range). (set $y = 0$)



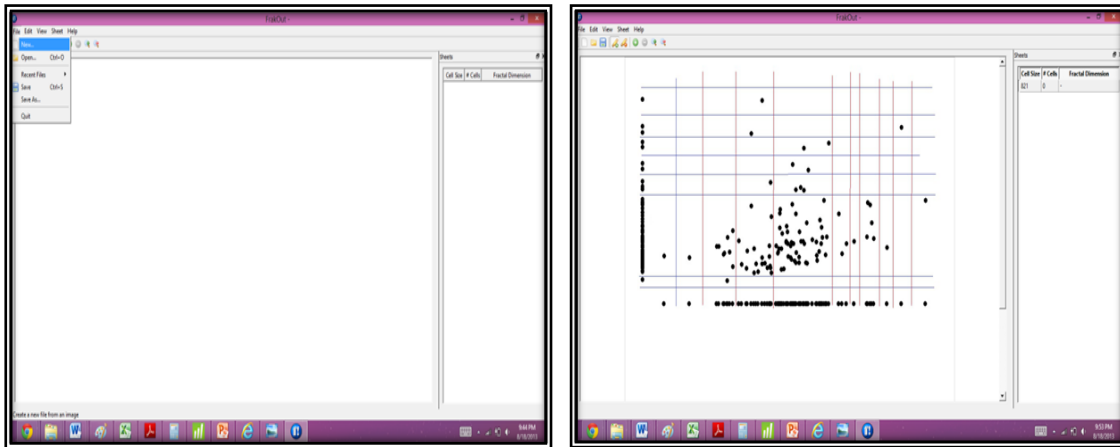
3. Project to y-axis and show ruggedness of y (amount of food). (set $x = 0$)



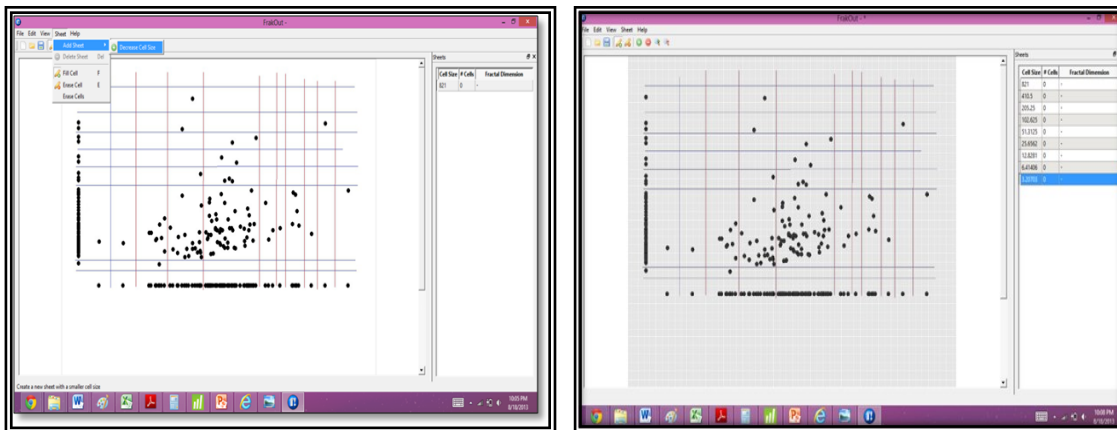
4. FRACTOGRAM: Combine x-y and scatterplot



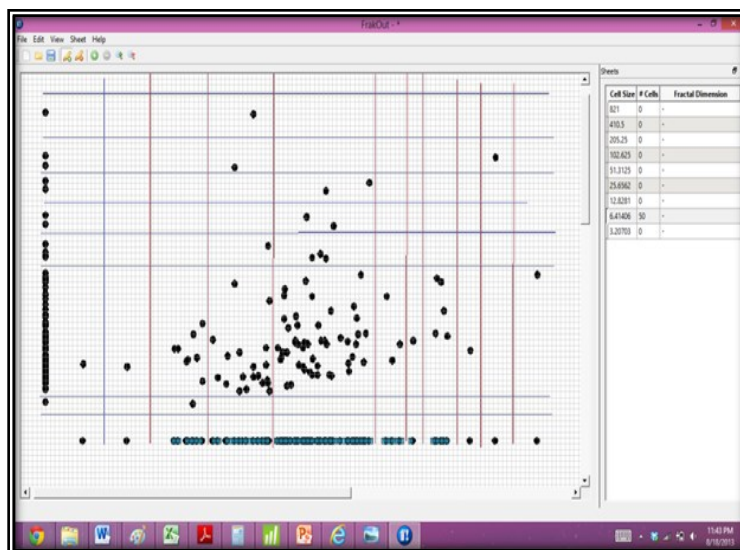
5. Open FrakOut.exe . Click File, New and Load the .jpg file.



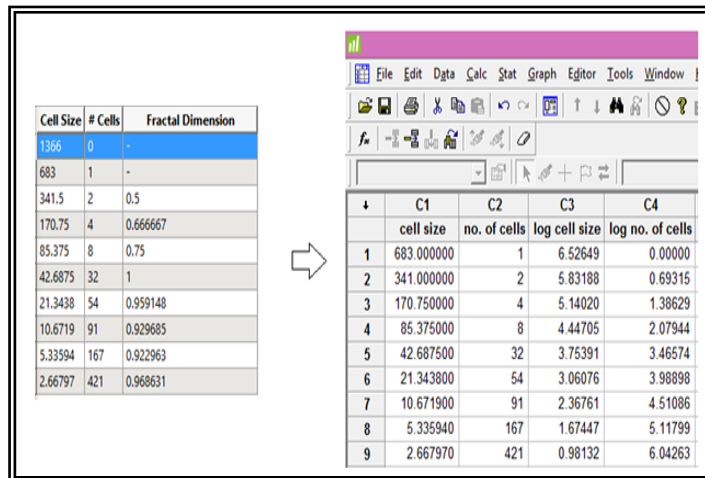
6. Click Sheet, Add Sheet and Decrease cell size.



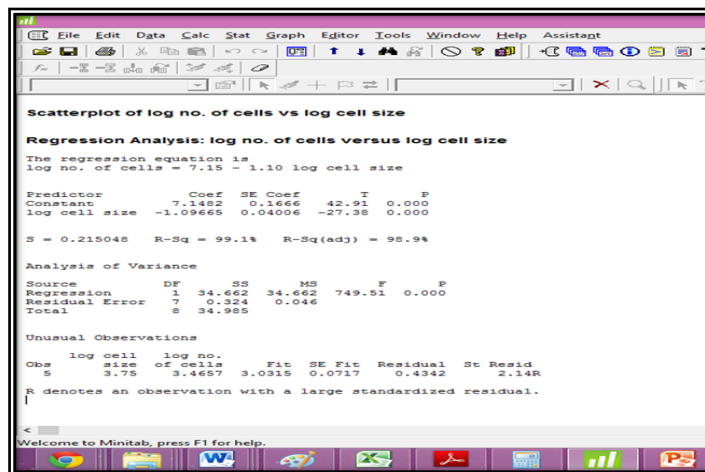
7. Connect and trace the points by highlighting the corresponding cells.



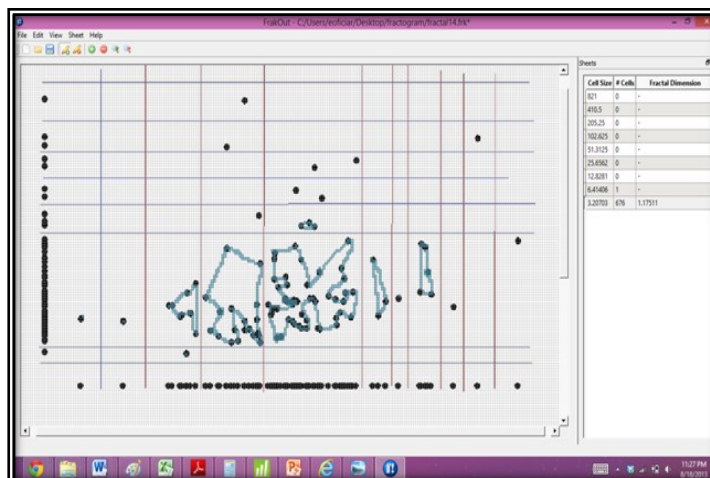
8. Go to EXCEL, copy the cell size and # of cells. Compute for the log (cell size). Compute for the log(# of cells).



9. Perform a linear regression analysis. The slope will give us the “fractal dimension” of the highlighted cells (ruggedness of x):



10. Connect and trace the points by highlighting the corresponding cell.



3.0 Simulation

We simulated the behavior of the proposed correlation measures (19) and (20) on three (3) cases:

1. Independent X and Y ($r_f = 0$)
2. Perfectly correlated X and Y ($r_f = 1$)
3. $0 < \text{corr}(X, Y) < 1$, $\text{corr}(x, y) = r_f = 0.6$

where, X and Y are the fractal random

variables having no Pearson product moment correlation measure. In order to simulate (c), we used Malik and Trudel's model (1986) represented by (15). For ease and simplicity, we used $n = 100$ repeated 1000 times. Table 1 shows the simulation results.

Table 1. Correlation of fractal random variables

Case	Roughness Correlation	Fractal Geometric Correlation	Assumed R_f
Independent X and Y	$R_f = 0.066$ p-value= 0.515	$R_f^h = 0.011$ p-value= 0.515	$R_f = 0.00$
Perfectly correlated X and Y	$R_f = 0.998$ p-value= 0.0001	$R_f^h = 1.000$ p-value= 0.000	$R_f = 1.00$
Highly correlated X and Y	$R_f = 0.682$ p-value= 0.000	$R_f^h = 0.646$ p-value= 0.000	$R_f = 0.63$

Note that the fractal geometric correlation (FGC) is closer to the assumed R_f .

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