Statistical Analysis of Fractal Observations: Applications in Education and in Poverty Estimation

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Abstract

We examine the power-law distribution with fractional exponents as a parent distribution of a random sample. The phase plot (x_t, x_{t+1}) of the random sample is shown to behave like fractals as defined by Mandelbrot (1967). Two (2) features of fractals are discussed as possible replacement of the mean (μ) and the standard deviation (σ) in describing fractals since the two quantities may not exist for low dimensional fractals $0 < d < 2$. The two (2) features are the quantiles and ruggedness measure via fractal derivatives and integrals.

Keywords: fractals, fractal derivatives, fractal integral, statistical fractals

1. INTRODUCTION

Scientific efforts to describe explain and predict nature and natural processes are hampered by the lack of fully-developed mathematical techniques to deal with massive data irregularities and ruggedness. Mathematical analysis assumes that the objects of study are smooth, linearly ordered and, in most cases, regular. Nature and natural processes, on the contrary, are rugged, irregular, discontinuous and often characterized by complex, non-linear interactions (Palmer, 1992). Mandelbrot (1967) suggested the use fractals for modeling such natural phenomena.

Fractal is a general term used to describe both exhibit self-similarity, scale invariance, fractional dimensions and heterogeneity (Mandelbrot the objects (geometry) and processes which exhibit self-similarity, scale invariance, fractional invariance, fractional dimensions and heterogeneity (Mandelbrot, 1982). These conditions are necessarily exhibited by all fractals in nature but they are not sufficient to completely define fractals. To date, a universally- accepted definition of a fractal has yet to be made but the absence of such a definition has not prevented scientists in varied disciplines to use the concept in applied work.

The use of fractals in ecology, biology and agriculture has been and is still actively pursued by scientists and researchers all over the world with relative success. Some of the problems successfully analyzed through fractals include: fractal dimensions of ecological landscape (Burrough, 1981); sustainable forestry (Crow, 1990); inter-specific competition in age-structured populations (Ebenman, 1987); dis-equilibrium silicate mineral textures (Fowler et al., 1987); forest geophysics (Khilmi, 1992); patterns of landscapes in disturbed environment (Krummel,1987); and others. Notable in all these studies is the preponderance of heuristics and ad hoc procedures due, perhaps, to the absence of a mathematical framework for fractal analysis.

This paper aims to discuss foundational

issues in statistical fractal analysis, present some new results and shed light on the strong connection between fractals and probability theory, as these apply to the study of social phenomena. We argue that most of the phenomena that had been modelled using the normal distribution can be more accurately analysed using statistical fractals since most of them exhibit self-similar stochastic patterns. The main advantage of using statistical fractal analysis over the classical normal distribution approach is that the former respects the inherent irregularities, ruggedness and stochastic self-similarity of natural phenomena while the latter tends to smooth out the values to conform to standard methods of statistical analysis using the normal curve.

Selvam (2011) succinctly describes the shift from normal distribution approach to statistical fractals as follows: *"The Gaussian probability distribution used widely for analysis and description of large data sets underestimates the probabilities of occurrence of extreme events such as stock market crashes, earthquakes, heavy rainfall, etc. The assumptions underlying the normal distribution such as fixed mean and standard deviation, independence of data, are not valid for real world fractal data sets exhibiting a scale-free power law distribution with fat tails. Fractal fluctuations therefore exhibit quantum-like chaos. The model predicted inverse power law is very close to the Gaussian distribution for small-scale fluctuations, but exhibits a fat long tail for large-scale fluctuations. Extensive data sets of Dow Jones index, Human DNA, Takifugu rubripes (Puffer fish) DNA are analysed to show that the space/time data sets are close to the model predicted power law distribution."*

2. Self-Similarity and Scale Invariance

Central to the study of fractals is the notion of self-similarity of an object at various scales. Horgan (1988) averred that fractals are geometric forms whose irregular details recur at different scales, that is, a fractal is a shape made of parts similar to the whole is some way (Mandelbrot, 1977). Self-similarity and scale-invariance, as described, can be translated mathematically as:

Definition 1: Let $f: V \to V$ where V is a vector space over the field R. If: $f(\alpha \upsilon) = \alpha^k f(\upsilon)$, $\alpha, k \in R^+$ then f is said to be scale-invariant or self-similar of order k.

In classical analysis, $k \in Z^+$ is a non-negative integer and f is called a homogeneous function of order k. However, Definition 1 allows for fractional orders. In fact, the study of fractals can be subsumed under a larger conformal symmetry analysis.

Theorem 1: If $V = R$, then the only scale invariant functions $f: R \to R$ are the power functions:

$$f(x) = cx^k, \; c \in R \quad \text{where } c = f(1).$$

We next define what we mean by a self-similarity dimension.

Definition 2: The fractal self-similarity dimension of an object having m copies of itself and scaled by a factor r is:

$$d = \frac{\log m}{\log r}$$

Thus, a regular square of unit side can be reproduced $m = 4$ times if we divide each side at the midpoint ($r = 2$). .A square will, therefore, have dimension:

$$d = \frac{\log 4}{\log 2} = \frac{2\log 2}{\log 2} = 2 \text{ as expected}$$

The Cantor set is the traditional representation of a fractal. It is obtained by dividing the closed interval [0, 1] into three and removing the middle third. The process is repeated on the two pieces [0, ⅓] and [⅓,1] by removing the middle third on the first piece and the middle third of the last piece and so on. The iterative process yields the set:

$$C = [0,1] \Big/ \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+1}{3^m}\right)$$

which looks like "fractal dusts". The dimension of the Cantor set is:

$$d = \frac{2\log 2}{\log 3} = 0.63 \text{ approximately}$$

Nature is replete with examples of fractals. To use an example from the forest, the perimeter of a maple leaf is not smooth; it is jagged. With video imaging system, Vlcek and Cheung (1986) generated a one-pixel-thick computer image of a leaf and found the fractal dimension to be d = 1.21 by comparing the log pixel length with the log number of lengths:

$$d = \frac{\log N\lambda}{\log \lambda}, \text{ where } \lambda = \text{pixel length}$$

The usefulness of the fractal concept stems from its ability to describe apparently random structures within a precise geometry (Orbach, 1987). The study of fractals, is, therefore, inextricably linked with statistical analysis. This exciting new area of statistics is now popularly known as statistical fractals.

3. Fractal Statistics

The study of geometric fractals naturally leads to the study of scale-invariant probability distributions *f(x)*. In particular, we restrict our attention to a random variable X whose support is non-negative and is scale-invariant. From Theorem 1, we know that *f(x)* has to take the form:

$$1\ldots\ f(x) = A x^{\lambda}, \quad \Theta < x < \infty$$

The particular power-law distribution of interest is given by:

$$2\ldots\ f(x) = \frac{\lambda-1}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda}, \ \Theta < x < \infty$$

The exponent of this power distribution corresponds to the fractal dimension of X. The corresponding cumulative distribution function of X is easily shown to be:

$$3\ldots\ F(x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}, \ \Theta < x < \infty$$

Power-law distributions are often used in practice when dealing with phenomena where there are smaller values then large values of X, e.g. income distribution more smaller values then large values of X, e.g. income distribution.

Let $x_1, x_2, \ldots, x_n$ be iid *F(x)*. A maximum likelihood estimator of λ is given by:

$$4\ \ldots.\ \hat{\lambda} = 1 + n\left[\sum_{i=1}^{n} \ln\left(\frac{x_i}{\theta}\right)\right]^{-1}$$

Alternatively, if we take the logarithm of both sides of Equation (2), we obtain:

$$5\ \ldots\ \log f(x) = C - \lambda \log \dot{x}$$

so that a plot of log *f(x)* versus log x yields a downward sloping line with slope λ. The slope of line (5) is, therefore, an estimate of the fractal dimension of X (or of the fractal object generated).

It is interesting to consider a practical situation where the observations actually come from a standard normal distribution N (0,1) . In this case:

$$5.1\ \ldots.\ f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2}, \quad -\infty < x < \infty$$

and so:

$$5.2\ \ldots.\ \log f(x) = \frac{-\log 2\pi}{2} - \frac{1}{2}x^2$$

An application of formula (4) for $\hat{\lambda}$ will show that but $\hat{\lambda} \rightarrow 1/2$ Equation (5) will not be linear with slope λ = 1/2. In other words, even if we obtained a fractional value of λ, this need not imply that the observation came from a power – law distribution or a fractal distribution

Next consider $x \underset{\sim}{d}$ exp (β=1), then

$$5.3\ \ldots\ f(x) = e^{-x},\ x > 0,$$

and:

$$5.3\ \ldots\ \log f(x) = -x$$

It follows that but $\hat{\lambda} \rightarrow 1$ but Equation (15) is not linear.

We note that in both instances, the probability distributions are not **scale-invariant** viz: $f(\alpha x) \neq \alpha^k f(x)$ for some k and $\forall \alpha$.

In fact:

Theorem 2: The only scale invariant probability distribution *f(x)* are those for which $y = x^{1-\lambda}$ is uniformly distributed where λ is the fractal dimension of *f(x)* .

Proof: Let *f(x)* be a scale invariant probability distribution of order λ, then

$$f(\alpha x) = \alpha^{-\lambda} f(x) = \frac{1}{\alpha^{\lambda}} f(x)$$

It follows that: $f(x \cdot 1) = \frac{1}{x^{\lambda}} f(1)$

or:

$f(x) = f(1) \cdot x^{-\lambda}$, a power – law distribution.

Let $y = \alpha^{1-\lambda}$ so $x = y^{\frac{1}{1-\lambda}}$. . The Jacobian of the transformation is the $J = \frac{1}{1-\lambda} y^{\frac{1}{1-\lambda}}$ distribution of y is:

$$g(y) = f\left(y^{\frac{1}{1-\lambda}}\right) \cdot \frac{1}{1-\lambda} y^{\frac{1}{1-\lambda}}$$

$$= \frac{f(1)\, y^{\frac{-\lambda}{1-\lambda}}\, y^{\frac{\lambda}{1-\lambda}}}{1-\lambda}$$

$$= \frac{f(1)}{1-\lambda} \cdot y^0 = \frac{f(1)}{1-\lambda} = k$$

Since *g(y)* is constant, it follows that y is uniformly distributed.

Theorem 2 is a good test to determine whether the probability distribution from which the observations came from is fractal or not.

Moments: Mean and Variance. Classical Statistics rely heavily on the mean to characterize the general behavior of a set of a set of data. Its use is justified on the basis of the fact that:

$$6 \ldots X_n \rightarrow \mu \quad \text{as } n \rightarrow \infty$$

where μ is the population mean. However, when the observations come from a power-law distribution, then X_n either continues to decrease with more observations i.e. $X_n \rightarrow -\infty$ as $n \rightarrow \infty$, or X_n continues to increase without bound with more observations, i.e. $X_n \rightarrow \infty$ as $n \rightarrow \infty$. The population mean μ does not exist when $\lambda < 1$. It will exist when $\lambda > 1$ but the variance σ^2 will not exist until λ reaches 2. The reliance to the Central Limit Theorem when using X as an estimator of μ will have to be carefully analyzed when sampling from real data.

Since the first two (2) moments of fractal distributions may not exist, we replace them with statistical descriptive measures that always exist. To this end, define:

Definition: Let $\delta_\lambda(x) = P(X \leq x)$ where $x \underset{\sim}{d} f(x;\lambda)$, is a fractal distribution with dimension λ . The α^{th} quantile of X is $x\alpha$, *and* $\delta\lambda\,(\tilde{x}_\alpha) = 1-\alpha$.

The α^{th} quantile of X always exists. It is the point $\tilde{x}_\alpha$ in the distribution such that (1—α) x 100% of the observations is below it. An explicit expression for $\tilde{x}_\alpha$ is:

$$(7)\quad \tilde{x}_\alpha = \theta\alpha^{\frac{1}{1-\lambda}}, \quad \theta < x < \infty.$$

In practice, we specify α and compute $\tilde{x}_\alpha$. The usual choice for α is $\alpha = ½$ or the **median**, but there is no particular reason why α should always.The fractal dimension λ describes how many smaller values there are than larger values in a fractal distribution. Thus, if $\lambda_1 < \lambda_2$, then there are more smaller values in $f(x;\lambda_1)$ than in $f(x;\lambda_2)$. This is particularly useful in applied work. For instance, by estimating the fractal dimensions of incomes λ_1 and λ_2, respectively in two (2) provinces, we can infer which of the two provinces have more “poor” than “non-poor” residents. If Province A has an income fractal dimension λ_1 while Province B has an income fractal dimension λ_2 and if $\lambda_1 < \lambda_2$, then Province A will have more poor residents than Province B. That is, the poverty incidence in Province A will be larger than the poverty incidence in Province B.

To see this more clearly, let $X \underset{\sim}{d} f(x;\lambda_1)$ and let $Y \underset{\sim}{d} f(y;\lambda_2)$ where $\lambda_1 < \lambda_2$. Then, the probability that an arbitrary x is less than an arbitrary y is:

(8)

$$P(x<y) = \int_\theta^\infty \left[\int_\theta^y \frac{(\lambda_1-1)}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda_1} dx\right]\left[\frac{(\lambda_2-1)}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda_2}\right] dy$$
$$= \int_\theta^\infty \left[\int_\theta^y \frac{(\lambda_1-1)}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda_1} dx\right]\left[\frac{(\lambda_2-1)}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda_2}\right] dy$$
$$= 1 + \frac{(\lambda_2-1)}{2-\lambda_1-\lambda_2}$$
$$= 1 - \frac{1-\lambda_2}{2-\lambda_1-\lambda_2}$$

while the probability that an arbitrary x is greater than an arbitrary y is:

(9) $$P(x>y) = \frac{1-\lambda_2}{2-\lambda_1-\lambda_2}$$

Suppose that $\lambda_1 = 0.2$ and $\lambda_2 = 0.5$ and , then $P(x<y) = 61.54\%$ which $P(x>y) = 38.46\%$ which i.e. it is more likely that an arbitrary *y* will be larger than an arbitrary x.

Distribution of the Sample Median. Let $x_1, x_2, \ldots, x_n$ be a random sample from F $(X;\lambda)$ and let and let $\tilde{x}$ = median $\{x_1, x_2, \ldots, x_n\}$. Then:

(10) $P(\tilde{X} \le \tilde{x}) = P$ (half of the observation are less than $\tilde{x}$)

$$= \binom{n}{\frac{n}{2}} P^{n/2}(1 \quad P)^{n-n/2}$$

Hence:

(11) $$G(\tilde{x}) = \frac{n!}{\left(\frac{n}{2}\right)!\left(n-\frac{n}{2}\right)!}\left[1-\left(\frac{\tilde{x}}{\theta}\right)^{1-\lambda}\right]^{n/2}\left(\frac{\tilde{x}}{\theta}\right)^{(1-\lambda)(n-n/2)}$$

We note that (11) is a Binomial distribution with parameter $\tilde{x}$ and n. by applying Slutsky's Theorem, we obtain:

(12) $\tilde{\mu} = E(\tilde{x})$ and $Var(\tilde{x}) = \frac{1}{4f^2(\tilde{\mu})}$ where $\tilde{\mu} = F_\lambda^{-1}\left(\frac{1}{2}\right)$

Ruggedness and Irregularities. A fractal object is characterized by its ruggedness and persistent irregularity in features yet self-similar Mathematically, this means that if the fractal is represented by a scale invariant function *f (x)* , then *f(x)* is continuous but nowhere differentiable. The notion "differentiability", therefore, needs to be re-examined.

The concept of a fractal derivative or fractional derivative can be used to describe the ruggedness of fractal features. Define

(13)… $D^n(f) = \frac{d^n(f)}{dx^n}$ $, n = 1, 2, \ldots, n$ to be the usual differential operator. Let $f(x) = x^k$ be a power law, it follows that:

(14) $$D(f) = \frac{d(x^k)}{dx} = kx^{k-1}$$
$$D^2(f) = k(k-1)\ \ x^{k-2}$$
$$D^\alpha(f) = \frac{k!}{(k-\alpha)!}\ x^{k-\alpha}\ , k > \alpha$$

If *k* and *α* are positive integers, then (8) is properly defined. In order to generalize to non-integral factorials, we define:

(15) $$D^\alpha(f) = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\ x^{k-\alpha} \quad k, \alpha \in R^+$$

where:

(16) $\Gamma(\alpha) - \int_0^\infty x^{\alpha-1}e^{-x}dx$. Equation (9) defines a fractal derivative which can be used to describe the "ruggedness" of a fractal. It can be shown that:

(17) $$D^\alpha(D^\beta)(f) = D^{\alpha+\beta}(f)$$

which implies that it is possible to decompose an entire fractal object and observe the resulting "ruggedness".

We examine the behavior of a random variable X from a power distribution with fractal dimension λ. From Equation (3), let:

(18)… $u = \left(\frac{x}{\theta}\right)^{1-\lambda}$ where $u \underset{\sim}{d} U(0,1)$.

It follows that:

(19) … $X = \theta\, u^{\frac{1}{1-\lambda}}$ whose fractal derivative can be obtained from (9) using $k = \frac{1}{1-\lambda}$ and $\alpha = \lambda$:

(20) $$D^\lambda(x) = \frac{\Gamma\left(\frac{1}{1-\lambda}+1\right)}{\Gamma\left(\frac{1}{1-\lambda}-\lambda+1\right)} u^{\frac{1}{1-\lambda}-\lambda}\cdot\theta$$
$$= \frac{\frac{1}{1-\lambda}\,\Gamma\left(\frac{1}{1-\lambda}\right)}{\frac{\lambda^2-\lambda+1}{1-\lambda}\,\Gamma\frac{\lambda^2-\lambda+1}{1-\lambda}} u^{\frac{\lambda^2-\lambda+1}{1-\lambda}}\cdot\theta$$
$$= \frac{\Gamma\left(\frac{1}{1-\lambda}\right)}{(\lambda^2-\lambda+1)\Gamma\frac{\lambda^2-\lambda+1}{1-\lambda}} u^{\frac{\lambda^2-\lambda+1}{1-\lambda}}\cdot\theta$$

If the fractal dimension is $\lambda = 1/2$, then:

(21) $\left|D^{\frac{1}{2}}(x)\right| \leq \frac{4\Gamma(2)}{3\Gamma\left(\frac{3}{2}\right)} - \frac{8\theta}{3\sqrt{\pi}} \simeq \pm 1.3108$ and if λ = ⅓, then

(22) $\left|D^{\frac{1}{3}}(x)\right| \leq \frac{81}{70} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{9}\right)} \leq \frac{81}{70} \simeq \perp 1.1579$

The higher the fractal dimension, the more rugged the features become.

In classical statistics, the information number I defined by:

(23) $I = \int_{-\infty}^{\infty} \left[\frac{f'(x)}{f(x)}\right]^2 f(x)dx$ is used to measure "dispersion". As a counterpart, we propose to use a "ruggedness" measure:

(24) $$I_R = \int_0^1 \left[\frac{\frac{d^\lambda x}{du}}{x}\right]^2 du$$

The "ruggedness" measure compares the fractal variation of x with the random variable x with respect to the uniform measure substituting (14) and (13) to (18) yields:

(25) $$I_R = k^2 \int_0^1 \left[\frac{u^{\frac{\lambda^2-\lambda+1}{1-\lambda}}}{\frac{1}{1-\lambda}}\right]^2 du$$

$$= k^2 \int_0^1 \left(u^{\frac{\lambda^2-\lambda}{1-\lambda}}\right)^2 du$$

$$= k^2 \int_0^1 u^{-2\lambda} du$$

$$= k^2 \left|\frac{u^{1-2\lambda}}{|1-2\lambda|}\right|_0^1 = \frac{k^2}{|1-2\lambda|}$$

where $k = \frac{\Gamma\left(\frac{1}{1-\lambda}\right)}{(\lambda^2-\lambda+1)\Gamma\left(\frac{\lambda^2-\lambda+1}{1-\lambda}\right)}$

We note that the "ruggedness" measure is a function of the fractal dimension λ. It describes how the irregularities in one scale are repeated in other scales. Equation (19) can be estimated from data when treated as a functional of the underlying F:

(26) $$I_R = \frac{k^2}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{2\lambda^2-2\lambda+1} = \frac{k^2}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{2\left(\lambda-\frac{1}{2}\right)^2} \cdot \sqrt{\frac{x_i}{\theta}}.$$

The table below compares the treatment of classical statistics and fractal statistics.

Table1. Data invariants

Statistical Analysis	**Characteristics**	**Spread**
A. Classical (Normal Dist.)	Typical or Average: μ	Standard Deviation: σ
B. Fractal Statistics	Feature or Fractal dimension: λ	Ruggedness Index: I_R

4. Applications in Real –World Modelling

We illustrate the applications of fractal statistics in two (2) settings: in Education, and in Poverty Estimation.

Education Setting: Test scores are often assumed to obey a normal distribution. For this reason, norm-referenced grading systems are based on the mean (and standard deviation). However, in reality, there will be more smaller scores than larger scores (especially in Mathematics classes). We are interested in modelling typical College Algebra classes in terms of their final examination scores.

In the **first setting**, the classes have the same fractal dimensions for their final examination scores. We generated thirty (30) observations from each of the classes (using λ = 0.67) and obtained estimates of the common fractal dimension. The results are shown in Table 2.

Table 2: Estimates of the fractal dimensions for Six (6) college Algebra classes

Class	**Estimate of Fractal Dimension with N=30**
1	0.670619
2	0.699700
3	0.652898
4	0.605367
5	0.702470
6	0.663639
Average Fractal Estimate	0.665782
Standard Deviation	0.03561

Note that the average fractal dimension estimate for the six classes is very close to the true fractal dimension of 0.67.

The effect of the sample size on the estimate of the fractal dimension is shown in Table 3:

Table 3: Effect of sample size on the fractal dimension estimate

Sample Size	Estimate of Fractal Dimension	Standard Deviation
30	0.665782	0.0361
60	0.685831	0.0330
90	0.672024	0.0201
120	0.657534	0.0199
240	0.671502	0.0110

We note that the convergence to the true fractal dimension is not uniform.

The results show that if the classes have the same fractal dimensions for their test scores, then the median score will be more or less the same for all the classes. Thus, instead of using the mean, the median will become the basis for grading the students in all the six classes.

In the **second setting,** we determine what happens when the fractal dimensions of the test scores are different for the six classes. In other words, we are investigating the effects of different "fractalities" when combined as one set of observations. We generate 30 observations each for the six classes of fractal dimensions $\lambda =$ 0.30, 0.40, 0.50, 0.60, 0.70, 0.80 (note that the first classes have more larger scores than any of the remaining five classes). Table 4 shows the results of the simulation.

The average of the fractal dimensions is 0.49998 which is not the same as the fractal dimension of the aggregated class (λ = 0.581064). The mean tends to smooth out the values of the test scores whereas the fractal dimension preserves the inherent "ruggedness" and "and "irregularities" of the test scores. The median scores for the six (6) classes and the median of the aggregated class are shown in Table 5.

Table 4. Six (6) classes of different fractal dimensions and the fractal dimension of the aggregated class

Class (N = 30)	**Estimated Fractal Dimension**	**Standard Deviation**
1	0.221184	0.031
2	0.332443	0.032
3	0.443826	0.034
4	0.554962	0.035
5	0.666333	0.033
6	0.777531	0.035
Aggregated Class (N = 180)	0.581064	0.010

Table 5. Median scores of the six classes and the aggregated class

Class	**Median Score**
1	44.94
2	52.43
3	62.93
4	78.63
5	104.89
6	157.30
Aggregated Class	83.55

The six classes had been set up so that the poorer students are placed mostly in the first class and the better ones in the sixth class. Taking the aggregated median score as the benchmark for all the six classes would favor the 5^{th} and the 6^{th} classes while marginalizing the first four classes. Meanwhile, if we consider the median of the median scores (62.93 + 78.63)/2), we would obtain a more realistic benchmark value of 70.78 for the aggregated class.

The rationale for using the median of the median scores as a benchmark value can be stated this way. For the individual classes, we decided to take the 50^{th} quantile (median) as a reasonable location parameter because the observations come from a fractal distribution.

When the classes are merged, the medians will preserve the "fractality" of the original observations, hence, it is logical to obtain the 50th quantile (median) of the merged fractal observations.

Poverty Estimation Setting. Poverty measurement remains an active area of research in the social sciences. This interest stem from the inclusion of poverty reduction (and hence, of poverty monitoring) as one of the Millennium Development Goals (MDG) for 2015 (United Nations Development Programme (UNDP), 2010). Alkire (2010) suggested the use of a Multidimensional Poverty Index or MPI which measures deprivation across Health, Education and Standard of Living. The MPI measures a different aspect of poverty from the usual income-based poverty index. Balicasan (2011) averred that there is a need to re-examine the weights applied to the different poverty dimensions (Health, Education and Standard of Living) in order for the MPI to be more reflective of the realities in the field. It will be noted that a poverty monitoring index has to be simple (easy to compute) and comprehensive in order for it to be widely – accepted for a cross – country comparison of poverty situations. The use of fractal indices of poverty satisfies this entire criterion and has the potential for wide applicability in the field of poverty estimation.

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