

Modelling In-service Teachers' Mathematical Reasoning in Constructing and Validating Proofs

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Abstract

This study determines the mathematical practices of proof validation and proof construction of the in-service teachers. Currently, little attention in the literature about how one's ability to validate correlates to the ability to construct proofs. Thus, we intend to fill this gap by providing a model about the mathematical proof reasoning of the 17 in-service teachers and to look into the connection between validation and construction of proofs using Toulmin's paradigm of reasoning. Data were gathered from interviews and written responses through worksheets. The data analysis was grounded in the common categories that emerged which highlights uncertainty of their possessed skill when encountered difficulties which require many resources to be proficient in both validation and proof construction. It shows that proof construction is more challenging than proof validation. Thus, the data that we obtained suggest that, if an individual can successfully validate purported proofs then it is not an assurance that he can also construct proofs.

Keywords: grounded theory, mathematics education, proof construction, proof validation, Toulmin's model

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1.0 Introduction

The role of proof in mathematics education has significantly contributed to the higher order thinking skills of every individual. The students must understand that proof construction and validation are at the heart of the creation and propagation of new mathematics (Janelle, 2014). Validating proofs is fundamental to proof construction. Constructing or producing proofs is inextricably linked to the ability to validate them reliably (Selden and Selden, 2003). To this extent, the skills on proving must be integrated into the mathematics curriculum at all levels. It was emphasized that reasoning and proving should always be part of the K- 12 mathematics classroom and any related proof activities must always incorporate in the learning experiences of the students at all times (NCTM, 2009). This suggests that the absence of "mathematical proof and reasoning" makes mathematics incomplete (NCTM, 2000). In fact, proving is a crucial process that needs major attention especially in improving students' understanding of mathematical arguments and proof construction. So there is a need to concentrate on enhancing the critical minds of the students.

As the NCTM (2000) emphasizes, the teacher plays a critical role in the teaching and learning process. Their experiences and knowledge towards the subject, the students are able to augment their understanding, their ability to solve problems, their confidence and attitude towards mathematics.

Recently, there were several studies about the students' conceptions on proofs and their abilities to construct proofs (Janelle, 2014; Basturk, 2010; Knuth, 2002; Thompson, 1984), but still, there is substantial evidence that most of the students encountered difficulties in constructing mathematical proofs. Even the university students cannot formulate a mathematical argument by their own (Mariotti, 2006). This is alarming considering that all students in tertiary level are expected to be knowledgeable in dealing with proofs.

Both Pfeiffer (2010) and Selden and Selden (2003) have reported that undergraduate mathematics students often have limited views of proof and fail to correctly validate arguments. Another literature shows that the students' method of proving was based on imitative reasoning by looking at a textbook as

their guide or through memorizing the proof algorithm (Basturk, 2010).

In 2012, it is interesting to note that the Department of Education of the Philippines embraces the K-12 curriculum, meaning the student must undergo the Kindergarten plus 12 years to complete the Basic Education Program. The realization of this curriculum was due to the poor performance of the Filipino students as reflected in their achievement scores in the National Achievement Test and the international test known as the Third International Mathematics and Science Study (TIMSS) (Tatsuoka, Corter, & Tatsuoka, 2004). Also, it is hard to hear that the Philippines was ranked as the lowest three among the 38 countries who participated in the Third International Mathematics and Science Study-Repeat (TIMSS-R).

There are many considerations that are essential that may lead to discerning the reason behind this scenario. Researchers revealed that one of the reasons behind is the value of proof and proving during elementary and high school which were not given emphasis in the process of teaching and learning mathematics. As a result, students including teachers could not validate arguments and construct proofs (Janelle, 2014). The elementary and secondary school teachers are struggling in teaching proofs since during their high school and college days; they were not exposed to proof construction. So, therefore, they are not prepared and fully equipped in implementing activities that improve the reasoning skills of the students (Knuth, 2002).

The intention of this study is to ascertain why most of the mathematics teachers are still struggling in proving and probably this study will help them to circumvent their difficulties in proving. Moreover, only a few researches are currently known about how the ability to validate relates to the ability to construct proofs and it was not fully understood (Janelle, 2014; Pfeiffer, 2010; Selden and Selden, 2003).

Hence, the goal of this paper is to create a model for in-service teachers' mathematical reasoning in constructing and validating proofs using Toulmin's model (Toulmin, 1958). The study concentrated on the proving and reasoning skills of the in-service senior high teachers because they are the prime provider and chief source in shaping the mind of the students

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inside the classroom. Also, this study sought to answer if there is a relationship between the ways in which the in-service teachers construct proofs and the ways in which they validate mathematical arguments. To address this question, we assessed and evaluated their proficiency in validating and constructing proofs through the use of Toulmin's model.

2.0 Methodology

Participants

There were 17 in-service senior high school mathematics teachers selected from different districts of Surigao City Division, Caraga Region. All of them were currently pursuing their master's degree program-their years of teaching experience varied from 2 to 20 years. Although they differ in their teaching experiences, they had previously taken and encountered mathematical proof. It is worth noting that the participants are interested to join the professional development because they are committed to reform their instructional practices. On the other hand, this will connect with the vision regarding the practice of teaching set forth by the Philippine K-12 curriculum.

Data Collection

A three- day seminar workshop was conducted at Surigao State College of Technology, Surigao City last January 14-16, 2017. In the said event, Toulmin's model was first introduced to the participants for them to be guided in validating the purported proofs and constructing proofs. The purported proofs were identical to those used by Selden and Selden (2003) and Weber (2008). Each participant was asked to write their comment in every line of the purported proof and decide whether the proof was valid or invalid. In construction of proofs, the participants were obliged to follow the scheme of Toulmin's model (Data, Warrant, and Conclusion) which serves as the scaffold so that construction becomes systematic and traceable.

To examine and compare the ways in which in-service senior high school mathematics teachers validate and construct mathematical arguments an interviewed data were collected. Each interview consisted of five questions, these include: *How would you define a mathematical proof? What is the purpose of proof in Mathematics? What are your methods or strategies use in validating and constructing a mathematical proof? What are the difficulties you encountered? What will you do to overcome these difficulties?*

The interview data were collected after the seminar-workshop to provide insight into the reasoning behind their validations and constructions. The outputs of these data are used to develop a model on the mathematical proof reasoning of the in-service teachers about the ways they validate proof and the ways they construct a proof.

Data Analysis

The data analysis was grounded through the use of Strauss and Corbin's (1998) coding methodology. The foundation of data analysis was processed through six stages of coding: semi-structured interviews, data transcription, open coding, axial coding, selective coding, and developing theory. The first stage is to gather qualitative data by conducting semi-structured

interviews. Then the data were organized through transcribing each interview. The transcripts were analyzed line- by- line that involves breaking down the data as the beginning step in coding and categorizing of phenomena.

The development and building of model about In-Service Teachers' Mathematical Reasoning in Constructing and Validating Proofs Using Toulmin's Model was anchored to the paradigm model introduced by Strauss and Corbin (1998). In exploring the relationships between categories, Strauss and Corbin recommended and defined the six basic components of the paradigm model (Causal Condition, Phenomena, Strategies, Contextual Condition, Intervening Condition, and Consequences).

3.0 Results and Discussions

Table 1 shows the participants' judgments on each of the proofs. Among the four purported proofs, only letter (b) is valid in which majority of them got it right. One of the participants, who decided that proof (a) is invalid, commented that he doesn't like the way the data was presented in the proof. Also, he found out that some of the arguments in the warrant don't hold when he applies counter examples. While one participant could not decide whether proof (a) is valid or not and three participants did not attempt since this is their first encounter to validate a proof. Most of the participants agree that proof (c) is valid and they admit that they forgot some of the concepts about divisibility. But one participant changed her mind that the proof is invalid. Furthermore, twelve participants decided that proof (d) is invalid. They noticed that there's something wrong with the data and the warrant is not convincing that needs further justification. They feel confused because they think that there is no connection between the data and warrant.

Table 1. Participants' judgments on each of the proofs

Proof Attempt	Valid Proof	Invalid Proof	Could not Decide	Did not Attempt
(a)	2	11	1	3
(b)	10	4	3	0
(c)	8	4	3	2
(d)	3	12	1	1

A summary of the frequency of each type of proof construction is provided in Table 2 below. With the use of Toulmin's model that serves as their guide, all participants had generated their own way of proving. Most of them established a complete and correct proof. Only a few had imperfectly created a proof because of their erroneous arguments while others can't systematically organize their thoughts.

Table 2. Participant's proof construction responses

Item No.	Something Wrong with the Data	Something Wrong with the Warrant	Complete & Correct Proof	Correct but Something Missing	Correct but not Systematic
1	3	2	6	6	0
2	2	0	10	2	3

It is alarming to note that majority of the in-service teachers have encountered great difficulty in constructing

mathematical proofs. They could not even write simple arguments to prove a theorem. Considering that one of the important skills to be acquired in mathematics is the ability to construct proofs. Despite its importance, they fail to create their own arguments because they could not use the syntactic knowledge that they had (Weber, 2001).

After transcribing the interviewed data, sixteen categories and fifteen categories were emerged from proof validation and proof construction, respectively. Also, there were nine

categories that are common to both. Using Strauss and Corbin (1998) paradigm model, the relationship of these categories were linked and generated a substantial theory entitled “A theory on In-Service Teachers' Mathematical Reasoning in Constructing and Validating Proofs” (fig. 1). The realization of this theory was anchored to the basic components of the paradigm model that are detailed in the diagram and elucidated below.

This theory illustrates the interplay of deep knowledge,

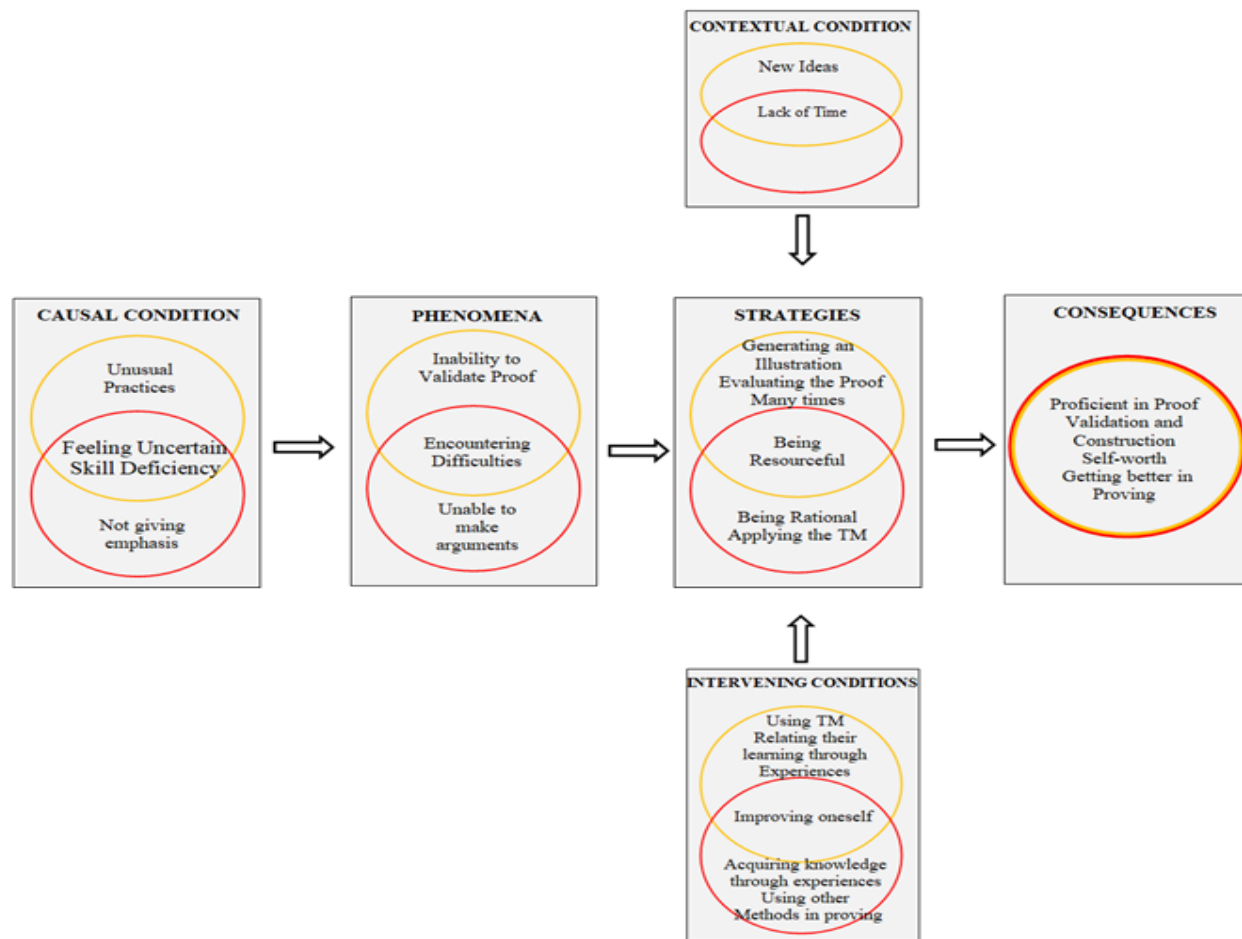


Figure 1. A theory on in- service teachers' mathematical reasoning in constructing and validating proofs

much practices, intention to persevere in the practice and finding the best resources for the purpose of improving self in the context of proof construction and validation. Passion and perseverance predicts success (Duckworth et al., 2009). Thus continuing the spaced out practice and aiming for advanced skills even during difficulties and uncertainties is much needed. To persevere in proof proficiency it should be intentional and much practice is needed to takes place and supplemented by finding the necessary resources available. Setting target is based on the resources and consideration of the desired cognitive processes and knowledge needed balanced by the intention to allocate time for the purpose of improving self (Willingham, 2009). However, feelings towards self-disposition, uncertainties, and how to manage difficulties should be a directed goal since this gives characterization of individuals' value system (Krathwohl, 2002).

Causal Conditions

The causal conditions represent the set of evidence that influences and enlighten the causes of difficulties of teachers towards proof validation and proof construction. The findings showed that there were many reasons that most of the teachers are still struggling and coping with understanding the mathematical proofs. Specifically, the teachers feel uncertain towards proof validation and proof construction because of their confusion, doubts, frustration, and lack of confidence in dealing with mathematical proofs. Another contributing factor is that they are not ready and fully equipped because of their insufficient knowledge and skills on the reasoning that they are not able to validate and construct a proof. These circumstances occurred because this is their first encounter in validating a proof. Also, they are not familiar with the use of Toulmin's model because they were not used to it. Furthermore, proof

construction was not given emphasis during their undergraduate math subjects.

The answer of Teacher 1, "*Wala nako magamit ako skills sa proving kay wala man nako gigamit sa klase og wala pa nako ma master an mga properties*" (I'm not used to, I didn't use these skills in everyday classes, and I do not master the properties) suggest that she does not have the sufficient time to practice from the basic to more advanced skills. Meanwhile, Teacher 4 answers on proof construction "*Sa tinod-anay, sa ako undergraduate, poor background gajud ko sa proof validation and proof construction kay gamay ra amo exposure ng proving*" (I actually have poor background on proof validation and proof construction on my undergrad, because I had little exposure on proving). Transfer will not happen, when experience and understanding do not go together (Prosser et al., 2005). Experience helps students see "deep structure and deep knowledge is hard-won and is the product of much practice" (Willingham, 2009 p.104).

Core Category

The core category identifies the incident that describes the core idea in which the relationships of all other categories were integrated. Generally, the current study shows that most of the teachers are encountering difficulties in validation and construction of proofs. For them, it is a complex process when identifying and completing the core elements of the Toulmin's model. They are struggling because they cannot recognize whether the proof is valid or not and they do not know how to start in creating a proof. In connection with these, they are not able to validate and construct proofs effectively. They tend to skip some parts of the proof that leads to the incorrect conclusion. Moreover, most of them were unable to make arguments when constructing proofs because they need more time to enhance their logical reasoning.

A note on teacher's response regarding the start of creating a proof (Teacher 7) "*Maglisod jud ko unsaon pagsugod sa ako proof, og unsa na mga methods ako gamiton*" this process is at the recognition stage "*maglisod pagsugod*"- locating the necessary steps needed for proof in order to proceed is a part of the cognitive process under remembering. While "*unsa na method*"- type of method is a type of conceptual knowledge. While, "*wala koy idea kon unsay nawong ng proof*" illustrates that a student is trying to figure out what the proof will look like, however, but failed to have a cognitive representation of the mathematical object and failed to have a logical conclusion drawn from the presented information. Now figuring out the proof of a mathematical statement is classified as a type of understanding conceptual knowledge.

Looking at the hierarchy of type of knowledge and cognitive process (Anderson, 2003; Krathwohl, 2002), remember conceptual knowledge, understanding conceptual knowledge is at the bottom of the hierarchy, while creating conceptual knowledge is at the top of the hierarchy. Because of the gaps of types of knowledge and cognitive process in the hierarchy between what students possess and what is required by the activity, converging the necessary prerequisite and the required knowledge might lead for students to progress in proof construction.

Strategies

These are the purposeful acts or strategies that give a solution to a problem in order to manage and respond the difficulties of the teachers towards mathematical proofs. Although most of the teachers are struggling in proving, but on the positive side, they are still doing their best to be resourceful in looking for ways and means to overcome their difficulties in validating and constructing proofs. They generate some illustrations using counter examples to elucidate the content of the proof and evaluate it for many times to ensure whether if it is valid or not. They also apply the Toulmin's model that serves as their guide that they become rational in constructing a proof. The following examples demonstrate how the teachers overcome their difficulties:

Teacher 1: "*Maghanap ko ng mga techniques og strategies sa proving*" (I will find more techniques and strategies on proving). "*Ako ini praktisan para ako mafamiliarize an mga methods sa proving*" (I will practice it so that I can familiarize the methods on proving).

Teacher 10: "*Magbasa ko og mga libro para ma widen pa ako knowledge sa proving*" (I will read books for me to widen my knowledge in proving). "*Ako pud gamiton ako mga old notes*" (I will also use my old notes).

Contextual Conditions

It refers to the location of events that is related to time, place, and how the participants reacted. It is based on their actions or interactions that lead to the creation of circumstances or problems that they encountered in validating and constructing mathematical proofs.

Some of the teachers do not have enough time in understanding the ways and techniques in validating and constructing a proof. They only encounter proof validation during their graduate studies. They are not even familiar with the important concepts needed in proving, considering that during their college days; proof validation and Toulmin's model was not introduced and emphasized. Some narratives are as follows:

Teacher 13: For me, there is no enough time to study on the techniques in validating proofs. Most of our time in school is devoted on solving problems in our respective subjects taught.

Teacher 17: Ahmmmm...because we are using the old curriculum in which proving was not emphasized.

Teacher 6: Toulmin's Model was kinda new to my vocab. It was my first time to encounter such word and method of validating and constructing of proof.

Intervening Condition

The intervening conditions are the conditions that provide the broad and general strategies in addressing difficulties of the teachers towards mathematical proofs that take place within a specific context. The result reveals that the teachers need to improve themselves to address their difficulties

towards proof validation and proof construction. They need to review and more practice on proving for them to master and gain further knowledge and techniques to enhance their proving skills. Using Toulmin model, which serves as their guide to provide a logical support, and relating their learning experiences also help them to validate a proof. They also use other methods, other than Toulmin's model, like two-column proof and finding patterns in constructing a proof that they acquire based on their previous experiences. Some examples are as follows:

Teacher 3: Learning to validate proofs is really a matter of continuous practice "learning- by- doing" and experience-based.

Teacher 7: But I believe with few more trial of using, I believe I will be able to master it and my skills in establishing proofs would be a lot better.

Teacher 12: On the other hand, the model also serves as a basis for structure and organization.

Consequences

It is the final stage of the paradigm model which is the result of strategies/actions in response to the difficulties of the teachers towards mathematical proofs. Overall, the results regarding consequences show that, when teachers had already attained the strategies/actions concerning to mathematical proof (validation and construction), they gain their self- worth that boost their confidence and satisfaction.

Teacher 9: Truly, knowing how Toulmin establish his proofs is totally helpful to have a more accurate and reliable proof.

Teacher 5: Applying Toulmin's Method of argumentation really helped me in structuring and organizing proofs.

Teacher 16: I could be able to construct proofs, I think. And boost my confidence in constructing some proofs.

4.0 Conclusions

The goal of this paper is to create a model for in-service teachers' mathematical reasoning in constructing and validating proofs using Toulmin's (1958) model. The emergent theory serves as evidence that proof validation is inextricably linked to proof construction. Most of the mathematics teachers are still struggling and encountering difficulties, in validating and constructing of proofs. They have skill deficiency and they feel uncertain towards it. They have insufficient knowledge and weak foundation on proving, thus they become doubtful, confused, and frustrated in dealing with it. In fact, using Toulmin's model, some of them cannot verify and justify if the given argument is valid or not. This problem arises because some of them have no enough time in enhancing and learning new ideas concerning with mathematical proof. As observed, except for those who can validate proof, they are struggling in creating their arguments in constructing a proof by their own. Thus, one can legibly construct proof if they can validate and

justify purported mathematical proof. Moreover, Selden and Selden (2003) propose that the explicit instruction in proof validation may improve students' abilities to validate and that this improvement might translate to greater success during proof construction.

Therefore, teachers should have the necessary skills in validating so that constructing of proof could be possible, however, it is not sufficient since the proficiency in proof declines or increase as they emerge in the practice of teaching mathematics as a function of choice to engage and not engage in process of construction, interaction with practice and self-interpretation of the role of proof in the mathematics as a whole.

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