Alternative Method for Finding Means of Fibonacci and Fibonacci-Like Sequences

[1]Romeo A. Patan and [1]Steven L. Elizalde

Abstract

Fibonacci sequence $\{F_n\}$ is defined as $F_1 = F_2 = 1$ and, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. F_n, is called nth Fibonacci number. When the first two terms of the Fibonacci sequence become arbitrary, it is known as Fibonacci-like sequence. Fibonacci-like sequence can start at any desired number. Natividad introduced a method for solving means of any Fibonacci-like sequence where it starts to find the first missing term of the sequence. This study derived an alternative method for finding means of any Fibonacci-like sequence. Descriptive case-investigatory and deductive process was employed to discover the alternative method with the utilization of the existing formulas and method of solving Fibonacci means. Based on the results of the study, it was concluded that the means of any Fibonacci-like sequence can be determined using the derived alternative method.

Keywords: Alternative Method, Fibonacci and Fibonacci-like Sequences, Fibonacci Means

Corresponding Author: *romeo_patan@yahoo.com*

1.0 Introduction

Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, ...) is a succession of numbers that are obtained through adding two preceding numbers. This sequence is named after Leonardo of Pisa who was known as Fibonacci (Natividad, 2011). There is also a sequence that is related to Fibonacci and it is called as Fibonacci -like sequence. The Fibonacci and Fibonacci-like sequences involve Fibonacci means. These means are related to arithmetic, geometric and harmonic means which are commonly tackled in college and high school Mathematics. In this study, the researchers aimed to introduce the existing method of solving Fibonacci means and to discover a new method for finding the means of Fibonacci and Fibonacci-like sequences. Problems included in this study focused on utilizing the existing method; and the use of the alternative method in finding the means of Fibonacci and Fibonacci-like sequence.

2.0 Research Methodology

This study utilized descriptive case-investigatory and deductive process to discover the alternative method for finding the means of Fibonacci and Fibonacci-like sequences. The data collected from the books and journals were examined in order to derive with an alternative method for solving means of any Fibonacci-like sequences. The researchers used the Binet's formula for finding the nth term of the sequence $\{F_n\}$ in deriving the formula for solving the last missing term of any Fibonacci-like sequence. After solving the last missing term $\{x_n\}$, the alternative method used the formula $F_n = F_{n+2} - F_{n+1}$ in finding the other means.

3.0 Results and Discussion

The Existing Method in Finding the Means of Fibonacci and Fibonacci-Like Sequences

Definition 1. Fibonacci numbers are the numbers in the integer sequence defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with $F_0=0$ and $F_1=1$ (Gulec and Taskara, 2009).

Using the formula above, it can be seen that the first few Fibonacci numbers are:

$$F_0=0;\ F_1=1;\ F_2=1;\ F_3=2;\ F_4=3;\ F_5=5;\ F_6=8;\ F_7=13;\ldots.$$

While the recurrence relation and initial values determine every term in the sequence, it would be nice to know the Binet's formula for $\{F_n\}$ so there is no need to compute all the preceding Fibonacci numbers in finding the nth term of the sequence.

Definition 2. The Binet's formula for finding the nth term of Fibonacci numbers $\{F_n\}$ is given by

$$F_n = \frac{1}{\sqrt{5}}[\phi^n - (1-\phi)^n]\ , \qquad (1)$$

Where ϕ is the golden ratio equal to $\frac{1+\sqrt{5}}{2}$ or 1.61803399… (Natividad, 2012).

Example 1: Find the 15[th] Fibonacci number using Binet's formula.
Solution:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}[\phi^n - (1-\phi)^n] \\ &= \frac{1}{\sqrt{5}}[1.618034^{15} - (1-1.618034)^{15}] \\ &= 610. \end{aligned}$$

Definition 3. If $a, x_1, x_2, \ldots, x_{n-1}, x_n, b$ is a Fibonacci sequence, then $x_1, x_2, \ldots, x_{n-1}, x_n$ are called Fibonacci means between a and b (Rabago, 2012).

Theorem 1. (Natividad's Formula) For any real number *a* and *b*,

$$x_1 = \frac{b - F_n a}{F_{n+1}}\ , \qquad (2)$$

Proof. See (Rabago, 2012) for the proof of this theorem.

By replacing the formula in (1) to the formula in (2), then the new formula for x_1 is given by

$$\begin{aligned} x_1 &= \frac{b - \frac{[\phi^n-(1-\phi)^n]}{\sqrt{5}}a}{\frac{[\phi^{n+1}-(1-\phi)^{n+1}]}{\sqrt{5}}} \\ &= \frac{b\sqrt{5} - [\phi^n-(1-\phi)^n]a}{[\phi^{n+1}-(1-\phi)^{n+1}]}\ , \qquad (3) \end{aligned}$$

[1]*Surigao del Sur State University-Main Campus*

Where x_1 is the first missing term in any Fibonacci-like sequence, a and b is the first and the last term given, n is the number of missing terms and ϕ = *1.61803399….*

Example 2: Insert six Fibonacci means between 1 and 21.
Solution:

a=1; b=21; n=6 and ϕ =1.618034.

$$x_1 = \frac{b\sqrt{5}-[\phi^n-(1-\phi)^n]a}{[\phi^{n+1}-(1-\phi)^{n+1}]}$$
$$= \frac{21\sqrt{5}-[1.618034^6-(1-1.618034)^6]1}{[1.618034^{6+1}-(1-1.618034)^{6+1}]}$$
$$= \frac{21\sqrt{5}-[1.618034^6-(1-1.618034)^6]}{[1.618034^7-(1-1.618034)^7]}$$
$$x_1 = 1.$$

Now that x_1 is already solved, then it is easy to find for x_2 which is

$$a + x_1 = x_2$$
$$1 + 1 = x_2$$
$$2 = x_2$$

The same method for x_3, x_4, x_5, x_6. Therefore the sequence is 1, 1, 2, 3, 5, 8, 13, 21, …which actually the Fibonacci sequence itself.

Using the Alternative Method in Finding the Means of Fibonacci and Fibonacci-Like Sequences

Using the new formula for x_1 in (3), the 1st mean of any Fibonacci-like sequence can be determined, so it can proceed in finding the other missing term/s. In this section, the researchers used another method of solving Fibonacci means. Instead of solving first the first missing term, the alternative method starts in solving the last missing term.

Definition 3. If a, x_1, x_2, …, x_{n-1}, x_n, b is a Fibonacci-like sequence, then x_1, x_2, …, x_{n-1}, x_n are called Fibonacci means between a and b (Rabago, 2012).

Corollary 1. For every natural number n,

$$x_n = \frac{F_n b-a}{F_{n+1}}, \quad (4)$$

Where n is the number of missing term(s), F_n is the nth Fibonacci number, a and b is the first and the last term of the sequence, respectively.

By replacing the formula in (1) to the formula in (4), then the new formula for x_n is given by

$$x_n = \frac{b[\phi^n-(1-\phi)^n]-a\sqrt{5}}{[\phi^{n+1}-(1-\phi)^{n+1}]}, \quad (5)$$

Where x_n is the last missing term in any Fibonacci-like sequence, a and b is the first and the last term given, n is the number of missing terms and ϕ =*1.61803399….*

Example 3: Sandywen wants to hybrid a sunflower with spirals. He found out that the inner spiral is 6 while the seventh spiral is 86. What are the Fibonacci numbers between them?
Solution:

a=6; *b*= 86; *n* = 5 and ϕ = 1.618034

$$x_n = \frac{b[\phi^n-(1-\phi)^n]-a\sqrt{5}}{[\phi^{n+1}-(1-\phi)^{n+1}]}$$
$$x_5 = \frac{86[1.618034^5-(1-1.618034)^5]-6\sqrt{5}}{[1.618034^{5+1}-(1-1.618034)^{5+1}]}$$
$$= \frac{86[1.618034^5-(1-1.618034)^5]-6\sqrt{5}}{[1.618034^6-(1-1.618034)^6]}$$
$$= 53.$$

Since the researchers applied the new formula, the formula $F_n = F_{n+2} - F_{n+1}$ was used instead of $F_n = F_{n-1} + F_{n-2}$ to find for the other means after solving the last missing term.

Solving for x_4:

$$b - x_5 = x_4$$
$$86 - 53 = x_4$$
$$33 = x_4$$

Solving all the missing terms using the alternative method brought out the Fibonacci-like sequence 6, 7, 13, 20, 33, 53, 86, ….

Example 4: Insert seven Fibonacci means between 2 and 47.
Solution:

a = 2; b = 47; n = 7 and ϕ = 1.618034

$$x_7 = \frac{47[1.618034^7-(1-1.618034)^7]-2\sqrt{5}}{[1.618034^{7+1}-(1-1.618034)^{7+1}]}$$
$$= \frac{47[1.618034^7-(1-1.618034)^7]-2\sqrt{5}}{[1.618034^8-(1-1.618034)^8]}$$
$$x_7 = 29.$$

Solving all the missing terms using the alternative method brought out the Fibonacci-like sequence 2, 1, 3, 4, 7, 11, 18, 29, 47, …. This sequence is a variant of Fibonacci sequence which is the Lucas sequence.

4.0 Conclusion

Based on the results of this study, the researchers concluded that the means of any Fibonacci-like sequence can be determined using the newly derived alternative method. It is also possible to find the means of Fibonacci and Fibonacci-like sequences in reverse way. The alternative method by the researchers greatly help for solving this type of means since it is very easy to use that even high school students can calculate it using basic algebra only.

Recommandations

This study encourages many researchers to discover their own method for solving means of the Fibonacci and Fibonacci-like sequences. However, the new alternative method can be used by the teachers and the students in solving Fibonacci means.

References

Gulec, H. H. & Taskara, N. (2009). On the properties of Fibonacci numbers with binomial coefficients. *International Journal of Contemporary Mathematical*

Sciences, 4(25), 1251-1256.

Natividad, L. R. (2011). On solving pell means. *International Journal of Mathematical Archive, 12(2), 2736-2739.*

Natividad, L. R. (2012). Fibonacci means and its applications. *International Journal of Mathematical Archive, 3, 1087 -1090.*

Rabago, J. F. T. (2012). On Natividad's formula for solving the missing terms of a recurrence sequence. *International Journal of Mathematical Archive, 8, 3105-3107.*